Multi-transmission-line-beam interactive system

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Abstract

We advance here the Pierce theory of travelling wave tubes by constructing a Lagrangian for a multi-transmission-line coupled to the electron beam (MTLB system). This theory yields in particular (i) the Euler-Lagrange equations governing the dynamics of the MLTB system, and (ii) exact expressions for the conserved energy and its flux distributions obtained from the Noether theorem. We identify in the Lagrangian a specific term associated with beam which is the origin of the mathematical mechanisms of amplification and the energy transfer from the beam to microwave radiation.

1 Introduction

The Pierce theory, [PierTWT], [Pier51, I], [SchaB, 4], is concerned with an electron beam interacting with surrounding it traveling tube. One of the theory motivations was to understand in the simplest terms the effect of energy transfer from the beam to the microwave radiation. The central part of Pierce's theory is the Pierce model which is based on waves interactions and consists of (i) an ideal linear representation of the electron beam and (ii) a transmission line (TL) representing the waveguide structure. This theory is the simplest one that accounts for wave amplification, energy extraction from an electron beam and its conversion into microwave radiation. With all its limitations the Pierce theory captures remarkably well significant features of wave amplification and the beam-wave energy transfer, and it is still in use for basic design estimates. To overcome the Pierce theory limitations far more sophisticated nonlinear theories have been developed to model very involved physics of the electron beam and slow-wave structures, [SchaB], [Gilm], [Tsim]. Needless to say that those theories are far more complex and often require a massive computer work.

In this paper we advance the Pierce theory to a theory that while keeping its simplicity and constructiveness allows for more complex slow-wave structures such as multi-transmission lines. We start with developing a field Lagrangian framework for the original Pierce model, and that already provides for a deeper understanding of the amplification and energy transfer mechanisms. We extend then that framework to a homogeneous and a nonhomogeneous system consisting of a multi-transmission line (MTL) coupled to the electron beam. We refer to such a system as MTLB system. We remind that general MTLs can approximate

with desired accuracy real waveguided structures which can be homogeneous (uniform) or inhomogeneous (nonuniform), [Nit], [Paul], [SchwiE].

The developed here Lagrangian field theory yields (i) the Euler-Lagrange field equations for a general MLTB system, and (ii) expressions for the conserved energy and its flux distributions obtained from the Noether theorem. We explore then in depth the mathematical mechanisms of the amplification and the energy transfer from the electron beam to the microwave radiation for general homogeneous MTLB systems. One of the novelties of the proposed here approach is that the analysis of those mechanisms bypasses entirely the electron bunching as a physical mechanism of the amplification but rather it relies on the model itself which captures the electron bunching in some form. Consequently, our analysis is a valuable source of a solid information on the electron bunching. The amplification phenomenon in general MTLB systems is studied by considering its exponentially growing eigenmodes and associated complex-valued wave numbers, just as in the original Pierce theory, [PierTWT], [PeirW, 11], [Pier51].

In the case of homogeneous MTLB system we carried out an exhaustive analysis of exponentially growing eigenmodes, their energy density and energy flux distributions as well as exact conditions providing for the amplification. The analysis includes derivation of a special canonical form of the dispersion relation having a very remarkable feature: one of its terms only carries information about the MTL, whereas the other only contains the beam parameters. Such separation drastically simplifies the amplification analysis.

The case of inhomogeneous MTLB systems is far more involved compared to homogeneous ones. In particular, for periodic MTLB systems the dispersion relations are not polynomial and that requires to turn to the most general form of the Floquet theory, [YakSta1, II, III]. In this case we derive the Euler-Lagrange field equations and transform them into the canonical Hamiltonian form using basics of the de Donder-Weyl theory, [Rund, 4.2]. This particular Hamiltonian form provides the basis for the most effective use of the Floquet theory, [YakSta1, II, III]. Detailed development of the Floquet theory for periodic MTLB system is very involved and left for another publication.

The structure of the paper is as follows: In Section 2 we briefly summarize our main results. Section 3 is devoted to the description of Pierce's model for beam-TL interaction as presented in [Pier51, I]. Section 4 deals with the Lagrangian approach to the model, including generalizations to both non-homogeneous and multiple transmission lines. In the following Section 5, we explore the amplification mechanism in the MTLB system as linked to instabilities in the dynamics of the beam. The appropriate mathematical setting, in particular the Hamiltonian structure of the model and its application to the study of eigenmodes in the most general case is the subject of Section 6 . In Section 7, we focus on the detailed study of growing modes for the homogeneous MTLB system. Section 8 deals with the questions of general energy conservation and energy transfer between the beam and the MTL on the growing mode. In Section 9 we make apparent how our general approach allows to easily recover some of the original Pierce's results.

A final section on mathematical subjects is included for reader's convenience. This section contains crashed theoretical expositions, some general computations and the mathematical details of some of the proofs, which have been deferred there to avoid distracting the reader from the flow of ideas in the main body of the paper.

2 Main results

One of the goals of this work is to identify the origin of the mathematical mechanism of amplification in MLTB systems. This goal has been accomplished by the construction of a Lagrangian field theory of MLTB systems that underlines their physical properties. Leaving detailed developments of this theory to the following sections we simply identify here the key term of the system Lagrangian responsible for amplification. This term quite expectedly is associated with the electron beam and is described by the following expression

$$\mathcal{L}_{\mathrm{b}}=\frac{\xi}{2}\left(\partial_t q+u_0 \partial_z q\right)^2, \qquad \xi=\frac{4\pi}{\omega_{\mathrm{p}}^2\sigma}>0, \tag{2.1}$$

where q is the space-charge ("smoothed-out jelly of charge", [Pier51, I]) flowing through the beam cross section σ, u_0 is the electron velocity and ω_{p} is the plasma frequency. $\mathcal{L}_{\mathrm{b}}$ *includes the beam potential energy* $\frac{\xi}{2}\left(u_0\partial_z q\right)^2$ *of positive sign, a marked feature distinguishing MLTB from common oscillatory systems in which the potential energy is always negative. The positive sign of this potential energy term is ultimately responsible for system instability and consequent amplification.* Indeed, a typical oscillatory system has a negative potential energy manifested in forces that move the system toward its equilibrium state. Such forces result in a stable motion with oscillatory energy transfer between its kinetic and potential forms. A qualitatively different picture occurs when the potential energy is positive. In this case resulting forces move the system away from the equilibrium at an exponentially growing rate. In its purest and simplest form such instability occurs for a mass point subjected to Hooke's force with a negative constant or its electric analog - a simple electric circuit with a negative capacitance. Interestingly, Pierce has observed an effective negative capacitance in his studies of a transmission line interacting with the electron beam, [Pier51].

Another marked feature of the term $\mathcal{L}_{\mathrm{b}}$ in (2.1) is its degeneration as quadratic form manifested as the perfect square expression. According to the general theory of unstable regimes, [YakSta1], this kind of degeneration is a necessary condition for instability arising under proper perturbations. From the point of view of the second order partial differential equation describing the beam dynamics this degeneracy is manifested as parabolicity compared to hyperbolicity occurring for common wave motion.

The power and efficiency of the Lagrangian approach is demonstrated by an exhaustive analysis of amplification regimes for a general homogeneous MTLB system, including precise conditions under which amplification takes place. In particular, if $0<v_1<v_2<\ldots<v_n$ denote the characteristic velocities of the MTL then we show that there is always an amplifying regime if $u_0\leq v_1$. If $u_0>v_1$ then amplification can occur only for sufficiently small ξ in (2.1). We also provide a transparent form of the dispersion relation for a general homogeneous MTLB system, including possible degenerations, as well as an asymptotic analysis of the amplification factor as $\xi\to 0$ and as $\xi\to\infty$. The case $\xi\to\infty$ corresponds to regimes considered by Pierce in [Pier51].

Yet another benefit of our Lagrangian approach is an exhaustive analysis of the energetic issues, including the overall energy conservation and energy transfer between the MTL and the beam. This analysis yields explicit expressions for the power $P_{\mathrm{B}\to\mathrm{MTL}}$ flowing from the beam to the MTL for an exponentially growing solution of the form

$$Q(z,t)=\widehat{Q}\mathrm{e}^{-\mathrm{i}(\omega t-k_0 z)}, \qquad q(z,t)=\widehat{q}\mathrm{e}^{-\mathrm{i}(\omega t-k_0 z)}, \qquad \operatorname{Im} k_0<0, \tag{2.2}$$

where Q is a coordinate describing the MTL and q is the one describing the beam. Namely, the following formula holds

$$\langle P_{\mathrm{B}\to\mathrm{MTL}}\rangle(z) = -\left[\frac{\omega\xi\,|k_0|^2\,|\widehat{q}|^2}{2}(\operatorname{Re} v_0 - u_0)\operatorname{Im} v_0\right]\mathrm{e}^{-2(\operatorname{Im} k_0)z}, \qquad v_0 = \frac{\omega}{k_0}. \tag{2.3}$$

We show that in the above formula the constant in front of the exponential is indeed *positive*, meaning that the energy flows from the beam to the MTL. Formula (2.3) indicates also that the power transferred to the MTL increases exponentially in the direction of the electron flow. The opposite is true of the evanescent wave when the power flows to the beam and decreases exponentially in the $+z$ direction.

3 Pierce's model

In [Pier51, I], J.R. Pierce presented a linear model for the description of charge-waves on an electron beam. The model is based on the following assumptions.

Assumption I. *Both the velocity and the current modulation on the beam (so called a.c. components) are small compared to the average or unperturbed velocity and current.*

This assumption justifies the linearization of the equations around the unperturbed regime. Let the total velocity of the electrons be $u_0 + v$, where u_0 is the average velocity and v is a small perturbation. Analogously, let $\rho_0 + \rho$ be the total electron density (per unit volume) where ρ_0 is the unperturbed density and ρ is the perturbation. Let σ be the cross section of the beam. Then, in the linear approximation, the total current flowing is $I_T = I_0 + I_{\mathrm{b}}$, where $I_0 = \sigma\rho_0 u_0$ is the d.c. current and the perturbation is given by

$$I_{\mathrm{b}} = \sigma\left(\rho u_0 + v\rho_0\right). \tag{3.1}$$

The linearized conservation of charge equation reads

$$\frac{\partial\rho}{\partial t} + \frac{\partial i}{\partial z} = \frac{\partial\rho}{\partial t} + \frac{1}{\sigma}\frac{\partial I_{\mathrm{b}}}{\partial z} = 0, \tag{3.2}$$

where i is the current density, $i = I_{\mathrm{b}}/\sigma$.

Assumption II. *The beam is thought of as a continuous medium (electron jelly) with no internal stress and a unique volumetric force acting along it.*

Its value per unit mass is given by $F = eE$, where E is an external electric field and $e = -|e|$ is the electron charge. Thus, it is assumed that the charge/mass ratio in the electron jelly is precisely e/m, m being the electron mass. The linearized motion equation for such a medium reads:

$$\frac{\partial v}{\partial t} + u_0\frac{\partial v}{\partial z} = \frac{e}{m}E. \tag{3.3}$$

Notice that in Pierce's original paper, [Pier51, I], the charge of the electron is denoted by $-e$, whereas here it is just e. Combining (3.1), (3.2) and (3.3) there follows

$$\partial_t^2 I_{\mathrm{b}} + 2u_0\partial_t\partial_z I_{\mathrm{b}} + u_0^2\partial_z^2 I_{\mathrm{b}} = \sigma\frac{e}{m}\rho_0\partial_t E. \tag{3.4}$$

Next, Pierce considers the interaction between a transmission line (TL) and the electron beam. The usual transmission line (telegraph) equations are modified by adding a source

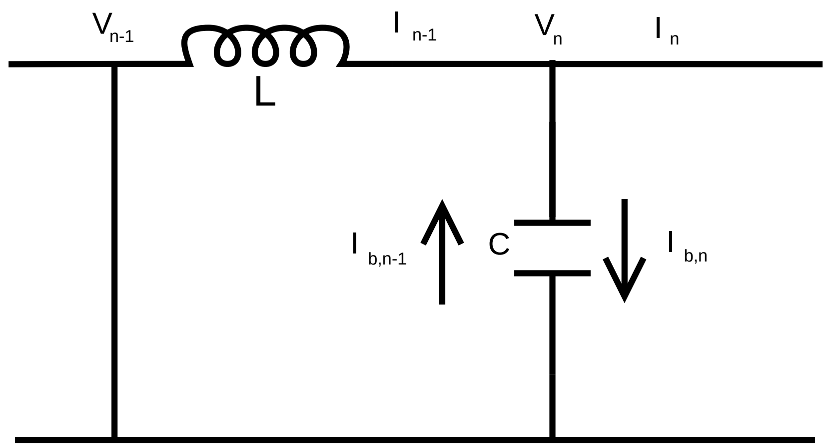

Figure 3.1: Discrete element of the TL-beam system in Pierce's model. The arrows represent shunt current induced on the capacitor.

term, which can be thought of as a shunt current instantaneously induced by the beam on the line, [Pier51, I],

$$\partial_z I = -C\partial_t V - \partial_z I_{\mathrm{b}}, \qquad \partial_z V = -L\partial_t I. \tag{3.5}$$

Here, as usual, $I = I(t, z)$ and $V = V(t, z)$ denote respectively the current through the inductive element and the voltage on the shunt capacitive element of the TL, $C > 0$ and $L > 0$ are respectively the shunt capacitance and inductance per unit of length. Note also that in the equations (3.5) $\partial_z I$ and $\partial_z V$ are respectively the current through the shunt capacitive element and the voltage drop on the inductive element of the TL per unit length. The addition of the source term $-\partial_z I_{\mathrm{b}}$ can be justified under the assumption of quasi-stationarity of the process: the charge wave on the beam "mirrors" onto the line. One of the lumped elements in the discretization of such excited TL is represented in Fig. 3.1. Induced current can be thought of as a distributed shunt current source.

As for the action of the TL upon the beam, it is governed by the axial electric field

$$E(t, x) = -\partial_z V(t, z) \tag{3.6}$$

acting upon electrons in the beam. Plugging the above expression into (3.4), we arrive at the equation

$$\partial_t^2 I_{\mathrm{b}} + 2u_0\partial_t\partial_z I_{\mathrm{b}} + u_0^2\partial_z^2 I_{\mathrm{b}} = -\sigma\frac{e}{m}\rho_0\partial_t\partial_z V. \tag{3.7}$$

Thus, according to [Pier51, I], the equations (3.5) and (3.7) constitute a model of the interactive TL-beam (TLB) system.

Some comments are in order. In more recent literature, improved versions of the linear Pierce model have been considered, see e.g. the chapter by J.H. Booske in [Nus]. These versions account for finer features such as bunching saturation, or retain the nonlinearity present in the original versions of equations (3.1) and (3.3), etc. Although such enriched models are undoubtedly more realistic and numerical computations based on them might provide a better agreement with experiment, they hardly allow for analytical treatment. In particular, they do not possess a Lagrangian structure. Pierce's model, though simple, already captures the mechanism of amplification and, as mentioned in the Introduction, can be generalized to the case of MTLB systems, and allows for a thorough mathematical analysis in all cases. Taking into account the fact that real wave guides can be approximated, in

principle, by an MTL with any degree of accuracy, [Nit], [Paul], [SchwiE], such generalization opens new perspectives in design optimization, which is the ultimate goal of our study.

4 Lagrangian formulation of Pierce's model

In this section we construct a Lagrangian field theory underlying the Pierce model. The Lagrangian theory provides a deeper insight into mathematical mechanism of amplification and energy transfer from the electron beam to the radiation.

4.1 The Lagrangian

The linear system of equations (3.5)-(3.7) arises as Euler-Lagrange equations associated to certain quadratic Lagrangian. To see this, let us first introduce the charge variables Q and q related respectively to the currents I and I_{b} by

$$I = \partial_t Q, \qquad I_{\mathrm{b}} = \partial_t q. \tag{4.1}$$

Thus the variables Q, q represent the amount of charge traversing the cross-section of the line (respectively the beam) at the point z within the time interval (t_0, t), where t_0 is some fixed reference time. Then the TLB system (3.5) and (3.7) takes the form

$$\partial_z Q = -CV - \partial_z q, \qquad \partial_z V = -L\partial_t^2 Q, \tag{4.2}$$

$$(\partial_t + u_0\partial_z)^2 q = -\frac{\sigma\omega_{\mathrm{p}}^2}{4\pi}\partial_z V, \tag{4.3}$$

where ω_{p} is the *plasma frequency* defined (in Gaussian units) by

$$\omega_{\mathrm{p}}^2 = \frac{4\pi e\rho_0}{m}, \tag{4.4}$$

[DavNP, 2.2].

Since it is not any harder to deal with inhomogeneous (in particular, periodic) TLs, we suppose from now on that C and L can be position dependent, that is

$$C = C(z), \qquad L = L(z). \tag{4.5}$$

Notice that the first equation in (4.2) readily implies the following representation for V

$$V = -C^{-1}\partial_z(Q + q). \tag{4.6}$$

Plugging the above expression for V into the second equation in (4.2) and into the equation (4.3) yield the following TLB evolution equations for the charges:

$$L\partial_t^2 Q - \partial_z\left[C^{-1}\partial_z\right](Q + q) = 0, \tag{4.7}$$

$$\xi(\partial_t + u_0\partial_z)^2 q - \partial_z\left[C^{-1}\partial_z\right](Q + q) = 0, \quad \xi = \frac{4\pi}{\omega_{\mathrm{p}}^2\sigma} = \frac{m}{\sigma e\rho_0} > 0. \tag{4.8}$$

We observe now that the above evolution equations are the Euler-Lagrange equations for the following Lagrangian

$$\mathcal{L}(z, \partial_t Q, \partial_z Q, \partial_t q, \partial_z q) = \frac{L}{2}(\partial_t Q)^2 - \frac{1}{2}C^{-1}(\partial_z Q + \partial_z q)^2 + \frac{\xi}{2}(\partial_t q + u_0\partial_z q)^2. \tag{4.9}$$

Indeed, for a general Lagrangian density $\mathcal{L} = \mathcal{L}(t, z; Q, \partial_t Q, \partial_z Q; q, \partial_t q, \partial_z q)$, the Euler-Lagrange equations take the form

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t Q)} + \partial_z \frac{\partial \mathcal{L}}{\partial(\partial_z Q)} - \frac{\partial \mathcal{L}}{\partial Q} = 0, \qquad \partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t q)} + \partial_z \frac{\partial \mathcal{L}}{\partial(\partial_z q)} - \frac{\partial \mathcal{L}}{\partial q} = 0 \ . \tag{4.10}$$

A straightforward computation confirms that the application of equation (4.10) to the Lagrangian defined by (4.9) indeed yields the TLB evolution equations (4.7) and (4.8). Pierce's original equations are obtained as a particular case, when C, L are constant along the line.

Let us make a final observation: It is assumed that the current induced by the beam onto the TL is due to the fact that the charge on the beam perfectly "mirrors" onto the waveguide. This assumption can be justified as an approximation in the "quasistatic" regime, in the spirit of Ramo's Theorem, [Ra]. According to some authors, e.g. R. Kompfner, [Kom] or J.H. Booske in [Nus], in dealing with real devices a coefficient $\varkappa \in (0,1)$ must be included in front of $\partial_z I_b$ in (3.5) (accordingly in (4.2)) to account for the real induced current, the case $\varkappa = 1$ being regarded as ideal. The Lagrangian approach can easily handle the general case. However, in order to keep the exposition as simple as possible, we only consider the ideal situation.

4.2 Generalization to multiple transmission lines

It is known that fairly general wave guides can be well approximated by multiple transmission lines, MTL. The corresponding generalization of Pierce's model is straightforward thanks to our Lagrangian formulation. Indeed, suppose that we have $n+1$ conductors, one of them being grounded, say the $(n+1)$-th. We denote by $V(z,t) = \{V_i(z,t)\}_{i=1\ldots n}$ the n-dimensional vector-column of voltages on the first n conductors with respect to the ground and by $I(z,t) = \{I_i(z,t)\}_{i=1\ldots n}$ the vector-column of currents flowing on them and set

$$Q(z,t) = \{Q_i(z,t)\}_{i=1\ldots n}, \qquad Q_i(z,t) = \int^t I_i(z,s)ds.$$

Let $L = L(z)$, $C = C(z)$ be the $n \times n$ matrices of self- and mutual inductance and capacity. As it is well known, they are positive symmetric (Hermitian). A natural generalization of (4.9) is provided by

$$\mathcal{L} = \frac{1}{2}\left\{(\partial_t Q, L\partial_t Q) - \left(\partial_z Q + \partial_z q B, C^{-1}\left[\partial_z Q + \partial_z q B\right]\right)\right\} + \frac{\xi}{2}\left(\partial_t q + u_0 \partial_z q\right)^2, \tag{4.11}$$

where (,) stands for the scalar product in $\Re^n$ and B is the n-dimensional vector-column with all components being the unity, that is

$$B = (1, 1, \ldots 1)^{\mathrm{T}} . \tag{4.12}$$

The corresponding Euler-Lagrange second order system is

$$\begin{gathered} L\partial_t^2 Q - \partial_z\left[C^{-1}(\partial_z Q + \partial_z q B)\right] = 0; \\ \xi\left[\partial_t^2 q + 2u_0 \partial_t \partial_z q + u_0^2 \partial_z^2 q\right] - \left(B^{\mathrm{T}}, \partial_z\left[C^{-1}(\partial_z Q + \partial_z q B)\right]\right) = 0. \end{gathered} \tag{4.13}$$

The generalized telegraph equations adopt the form

$$\partial_z I = -C\partial_t V - \partial_z I_b B; \qquad \partial_z V = -L\partial_t I. \tag{4.14}$$

Observe that the choice of the vector B assumes, besides perfect induction, a symmetry in the interaction between the beam and the different lines. A more realistic approach might include coefficients $\varkappa_i \in (0,1)$ in vector B to account for non-symmetric interaction. As we already mentioned in Subsection 4, such effects can be easily handled by our approach.

It should be mentioned that the above formal generalization to the case of several lines can be actually derived from Maxwell's equations under reasonable assumptions. See, for example, [Nit, 2], [Paul, 1.4.1] for models of interacting TLs.

5 Amplification analysis

Evidently the beam is a sole source of the energy in the system and, hence, it has to be related to the existence of system exponentially growing modes. In this section we analyze the mechanism underlying the exponential modes.

To trace the amplification to the beam we view the Lagrangian (4.11) as a perturbation of the Lagrangian $\mathcal{L}_{\mathrm{b}}$ for the isolated beam defined by

$$\mathcal{L}_{\mathrm{b}} = \frac{1}{2}\left(\partial_t q + u_0 \partial_z q\right)^2 = \frac{1}{2}\left[\left(\partial_t q\right)^2 + 2u_0 \partial_t q \partial_z q + u_0^2 \left(\partial_z q\right)^2\right]. \tag{5.1}$$

We introduce then the equivalent Lagrangian $\widetilde{\mathcal{L}} = \frac{1}{\xi}\mathcal{L}$, where $\mathcal{L}$ is as in (4.11), i.e.

$$\widetilde{\mathcal{L}} = \mathcal{L}_b + \varepsilon \mathcal{L}' = \frac{1}{2}\left(\partial_t q + u_0 \partial_z q\right)^2 + \tag{5.2}$$
$$+\frac{\varepsilon}{2}\left\{\left(\partial_t Q, L \partial_t Q\right) - \left(\partial_z Q + \partial_z q B, C^{-1}\left[\partial_z Q + \partial_z q B\right]\right)\right\},$$

and $\varepsilon = 1/\xi$. Small values of ξ defined by (4.8) and consequently large values ε correspond to strong coupling and regimes where the beam effectively feeds its energy into transmission lines in the form EM field. The EM field energy gain originates in the beam as an infinite reservoir of the potential energy $\frac{1}{2}(u_0 \partial_z q)^2$. Importantly, the potential energy enters the Lagrangian with the positive sign unlike in oscillatory systems. For small coupling as we will show no energy transfer might occur from the beam to the EM field. This perturbation analysis suggests to consider first the beam as an isolated system.

5.1 Charge wave dynamics

In this subsection, we investigate charge wave dynamics on the beam as an isolated system, described by (5.1). We already mentioned the role of the term $u_0^2\left(\partial_z q\right)^2$ as a source of energy. This term is responsible for the system instability manifesting itself by exponentially growing solutions of the associated E-L equations. The gyrotropic term $u_0 \partial_t q \partial_z q$ in the Lagrangian provides for stabilizing effect. As we will see, for the Lagrangian (5.1) the balance between instability and stability is struck exactly in the margin. Namely, a small perturbation of this Lagrangian can make the system either stable or unstable.

The beam Lagrangian $\mathcal{L}_{\mathrm{b}}$ is quadratic in $(\partial_t q, \partial_z q)$, see Section 10.2, and has the following structure

$$\mathcal{L}_{\mathrm{b}} = \frac{1}{2}\alpha(\partial_t q)^2 + \theta \partial_t q \partial_z q - \frac{1}{2}\eta(\partial_z q)^2 = (\partial_t q, \partial_z q)^{\mathrm{T}} M (\partial_t q, \partial_z q), \tag{5.3}$$

where

$$M = \begin{bmatrix} \alpha & \theta \\ \theta & -\eta \end{bmatrix} = \begin{bmatrix} 1 & u_0 \\ u_0 & u_0^2 \end{bmatrix}, \quad (\text{thus } \alpha = 1,\ \theta = u_0,\ \eta = -u_0^2). \tag{5.4}$$

The corresponding Euler-Lagrange equation (10.16) is

$$\left(\partial_t + u_0 \partial_z\right)^2 q = 0. \tag{5.5}$$

Applying the general formulas (8.3)-(8.4) for the energy H and its flux S we obtain

$$H_{\mathrm{b}}[q] = \frac{1}{2}\left(\partial_t q\right)^2 - \frac{u_0^2}{2}\left(\partial_z q\right)^2, \tag{5.6}$$

$$S_{\mathrm{b}}[q] = \partial_t q\left(u_0 \partial_t q + u_0^2 \partial_z q\right) = u_0 \partial_t q\left(\partial_t q + u_0 \partial_z q\right) = u_0\left(\partial_t q\right)^2 + u_0^2 \partial_t q \partial_z q. \tag{5.7}$$

5.1.1 Charge wave eigenmodes and stability issues.

Since the beam parameters are constant in space we can make use of the dispersion relation to study the eigenmodes. Thus, plugging $q(z,t) = \mathrm{e}^{-\mathrm{i}(\omega t - kz)}$ in (5.5), we get

$$\omega^2 - 2u_0 \omega k + u_0^2 k^2 = \left(\omega - u_0 k\right)^2 = 0,$$

hence $k_\omega = \omega / u_0$ is a double real root. The corresponding eigenmodes are $q_1(z,t) = \mathrm{e}^{\mathrm{i}(k_\omega z - \omega t)}$ and $q_2(z,t) = z\mathrm{e}^{\mathrm{i}(k_\omega z - \omega t)}$ or their real valued counterparts

$$v_1(z,t) = \cos\left(kz - \omega t\right), \qquad v_2(z,t) = z\cos\left(kz - \omega t\right).$$

The associated energy flux is, according to (5.7),

$$S_{\mathrm{b}}[v_1] = 0, \qquad S_{\mathrm{b}}[v_2] = -u_0^2 z\omega \sin\left(kz - \omega t\right)\cos\left(kz - \omega t\right). \tag{5.8}$$

To make useful inference related to conservation laws it is common to use the following time-averaging operation. Namely, for a (locally integrable) function f defined on $[0, \infty)$ we introduce

$$\langle f \rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T f(t)\, \mathrm{d}t. \tag{5.9}$$

This time-averaging operation has the following properties. If f is a smooth and bounded function on $[0, \infty)$, then

$$\left\langle \frac{df}{dt} \right\rangle = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{df}{dt}\, \mathrm{d}t = \lim_{T \to \infty} \frac{1}{T}\left[f(T) - f(0)\right] = 0; \tag{5.10}$$

The differentiation operation with respect to parameters commute with the time-averaging operation. Namely, if f also depends (smoothly) on some parameter z the following identity holds

$$\langle \partial_z f \rangle = \partial_z \langle f \rangle. \tag{5.11}$$

Since $\mathcal{L}_{\mathrm{b}}$ does not depend on time explicitly, conservation of energy takes place, (10.56):

$$\frac{\partial H_{\mathrm{b}}}{\partial t} + \frac{\partial S_{\mathrm{b}}}{\partial z} = 0.$$

Taking the average on both sides and using the above properties of averaging, we conclude that

$$\langle S_{\mathrm{b}}[v_2] \rangle (z) = \mathrm{const}.$$

On the other hand, it follows from (5.8) that $\langle S_{\mathrm{b}}[v_2]\rangle(0) = 0$. Hence $\langle S_{\mathrm{b}}[v_2]\rangle(z) = 0$.

From the stability point of view, this situation is a very degenerate one. To illustrate this point, let us consider a perturbed beam dispersion relation

$$\omega^2 - 2\alpha u_0 \omega k + u_0^2 k^2 = 0 \quad \text{or} \quad (\omega - \alpha u_0 k)^2 = \left(\alpha^2 - 1\right) u_0^2 k^2,$$

where α is a real number. This equation can be readily recast as

$$\omega - \alpha u_0 k = \pm\sqrt{\alpha^2 - 1}\, u_0 k,$$

and it has the following solutions

$$k_\omega(\alpha) = \frac{\omega}{u_0}(\alpha \pm \sqrt{\alpha^2 - 1}).$$

Notice that if $\alpha^2 < 1$ the above solutions become complex conjugate, whereas if $\alpha^2 > 1$ they are real distinct. $\alpha = 1$ corresponds to a double real solution, already showing the degeneracy.

An important subject of our interest is the analysis of MTL structures in which the parameters vary periodically in z. The Floquet theory and, in particular, the Floquet multipliers are the mathematical objects that deal with such situations *par excellence*. As explained above, we consider the coupled system as a perturbation of the beam. Consequently, it is instructive to take a look at the isolated beam in the light of Floquet theory with arbitrary period (eventually dictated by the period of the structure).

The Floquet multipliers with period unity are $\rho_\omega(\alpha) = \mathrm{e}^{\mathrm{i}k_\omega(\alpha)}$. It is clear that in the case $\alpha^2 < 1$ they are symmetrically located with respect to the unit circle in the complex plane. The solution corresponding to the multiplier outside the circle is a growing wave, whereas the one corresponding to the multiplier inside the circle is an evanescent one. In the opposite case $\alpha^2 > 1$, both roots are located on the unit circle, and the corresponding modes are purely oscillatory. We refer to these two qualitatively different perturbations as respectively unstable and stable. Aiming at amplification by coupling the beam to a MTL, the unstable situation is one to be favoured. The desired instability for spatially homogeneous MTL is achieved by sufficiently strong coupling (small values of ξ). An extension of this result to the case of periodic MTLs is left for a forthcoming publication.

Additional insight in mathematical mechanism of amplification associated with instability can be gained by looking at the nature of the partial differential equations involved. Indeed, the equation

$$\left(\partial_t^2 + 2\alpha u_0 \partial_t \partial_z + u_0^2 \partial_z^2\right) q = 0 \tag{5.12}$$

is hyperbolic if $\alpha^2 > 1$. In this case, there are two propagation velocities $v^\pm(\alpha)$ of the same sign, namely

$$v^\pm(\alpha) = \frac{\omega}{k_\omega^\pm(\alpha)} = -u_0(\alpha \mp \sqrt{\alpha^2 - 1}),$$

and the general solution has the form

$$q(z,t) = q_1(z - v^+ t) + q_2(z - v^- t).$$

Therefore, any solution which is bounded in time (as it is the case for harmonic in time solutions) is automatically bounded in space. In other words, no harmonic in time regime can be exponentially growing in space.

In the critical case, $\alpha = 1$, the equation is of the parabolic type. Changing variables $(z,t) \to (\xi, \eta)$ with $\xi = x - u_0 t$, $\eta = ax + bt$ with $b + au_0 \neq 0$, it can be easily checked that the general solution in this case is

$$q(z,t) = zF(z - u_0 t) + G(z - u_0 t) = t\widetilde{F}(z - u_0 t) + \widetilde{G}(z - u_0 t),$$

where $F, G, \widetilde{F}, \widetilde{G}$ are arbitrary functions. In particular any travelling wave with velocity u_0 is a solution. Again here we see that bounded in time dependence can be accompanied by at most linear growth in space.

If $\alpha^2 < 1$ we are dealing with the elliptic case where there is no propagation. This is the only case allowing for the exponential amplification. Indeed, a linear change of variables $(z,t) \to (\xi = az + bt, \eta = cz + dt)$ transforms the equation into the Laplace equation

$$u_{\xi\xi} + u_{\eta\eta} = 0,$$

which admits real solutions of the form $u(\xi, \eta) = \mathrm{e}^{k\xi} \cos(\omega\eta)$, $u(\xi, \eta) = e^{k\xi} \sin(\omega\eta)$ etc.

The considered above artificial perturbation of the beam equation was for illustration purposes to see the degenerate stability properties of the system under perturbation of its parameters. For the MTLB system, however, it is the MTL the one that plays the role of perturbation. In Section 7, we prove that, when the beam is properly coupled to an homogeneous MTL, this delicate equilibrium is broken in favor of an unstable regime and resulting amplification.

6 Hamiltonian structure of the MTLB system

In order to study the MTLB system, in particular the associated modes, their stability and the amplification phenomenon, we make use of the Hamiltonian structure associated to the Lagrangian (5.2). More precisely, we use a version of Hamiltonian formalism that treats the space and time variables on the same footing, known as de Donder-Weyl formalism. For reader's convenience, we have gathered the basic information about this topic in Section 10.1. As usual, the passage from Lagrangian to Hamiltonian point of view allows to cast the second-order Euler-Lagrange system of equations (4.13) in the form of a first order system, either with respect to t or with respect to z.

To comply with notations of Section 10.1, from now on we put $\mathsf{q}_1 = Q$, $\mathsf{q}_2 = q$, $\mathsf{q} = (\mathsf{q}_1, \mathsf{q}_2)$. The Lagrangian $\widetilde{\mathcal{L}}$ in (5.2) is quadratic in its variables $(\partial_t Q, \partial_t q, \partial_z Q, \partial_z q)$, that is in $(\partial_t \mathsf{q}, \partial_z \mathsf{q})$ in the new notation. Indeed,

$$\widetilde{\mathcal{L}} = \frac{1}{2} \partial_t \mathsf{q}^{\mathrm{T}} \alpha \partial_t \mathsf{q} + \partial_t \mathsf{q}^{\mathrm{T}} \theta \partial_z \mathsf{q} - \frac{1}{2} \partial_z \mathsf{q}^{\mathrm{T}} \eta \partial_z \mathsf{q}, \tag{6.1}$$

where

$$\alpha = \begin{bmatrix} \varepsilon L & 0 \\ 0 & 1 \end{bmatrix}, \quad \theta = \begin{bmatrix} 0 & 0 \\ 0 & u_0 \end{bmatrix}, \quad \eta = \begin{bmatrix} \frac{\varepsilon}{C} & \frac{\varepsilon}{C} \\ \frac{\varepsilon}{C} & \frac{\varepsilon}{C} - u_0^2 \end{bmatrix}, \tag{6.2}$$

or when using block matrix,

$$\widetilde{\mathcal{L}} = \frac{1}{2} \mathsf{u}^{\mathrm{T}} M_{\mathrm{L}} \mathsf{u}, \quad \text{with} \qquad M_{\mathrm{L}} = \begin{bmatrix} \alpha & \theta \\ \theta & -\eta \end{bmatrix}, \qquad \mathsf{u} = \begin{bmatrix} \partial_t \mathsf{q} \\ \partial_z \mathsf{q} \end{bmatrix}. \tag{6.3}$$

The derivation of de Donder-Weyl version of Hamilton equations in the variable z for quadratic Lagrangians is described in Section 10.2. In particular, for $\widetilde{\mathcal{L}}$ defined by (6.1) the Hamiltonian $H_{\mathrm{DW}}(\mathsf{p})$ does not depend on q and

$$H_{\mathrm{DW}}(\mathsf{p}) = \widetilde{\mathcal{L}}(\mathsf{u}), \text{ where } \mathsf{p} = M_{\mathrm{L}}\mathsf{u},$$

$\mathsf{p} = (\mathsf{p}_t, \mathsf{p}_z)$ being the vector of momenta or, or equivalently,

$$H_{\mathrm{DW}}(\mathsf{p}) = \frac{1}{2}\mathsf{p}^{\mathrm{T}} M_{\mathrm{L}}^{-1}\mathsf{p}. \tag{6.4}$$

In the variables $(\mathsf{p}_z, \partial_t \mathsf{q})$, the first-order system reads

$$\tilde{J}\partial_z V = \mathrm{i}\partial_t \tilde{M} V, \qquad V = \begin{bmatrix} \mathsf{p}_z \\ \partial_t \mathsf{q} \end{bmatrix}, \tag{6.5}$$

where

$$\tilde{J} = \begin{bmatrix} 0 & \mathrm{i}\mathbf{1} \\ \mathrm{i}\mathbf{1} & 0 \end{bmatrix}, \qquad \tilde{M} = \tilde{M}(z) = \begin{bmatrix} -\eta(z)^{-1} & \eta(z)^{-1}\theta \\ \theta\eta(z)^{-1} & -\alpha(z) - \theta\eta(z)^{-1}\theta \end{bmatrix}. \tag{6.6}$$

Observe that $\tilde{J}$ and $\tilde{M}$ are respectively antihermitian and hermitian, that is

$$\tilde{J}^* = -\tilde{J}, \qquad \tilde{M}^* = \tilde{M}.$$

Consider now a time harmonic solution

$$V(z,t) = \begin{bmatrix} \mathsf{p}_z \\ \partial_t \mathsf{q} \end{bmatrix} = \hat{V}(z)\,\mathrm{e}^{-\mathrm{i}\omega t}, \qquad \hat{V}(z) = \begin{bmatrix} \widehat{\mathsf{p}}_z(z) \\ -\mathrm{i}\omega\widehat{\mathsf{q}}(z) \end{bmatrix} \tag{6.7}$$

to the Hamiltonian equation (6.5), which is reduced to

$$\tilde{J}\partial_z \hat{V} = \omega\tilde{M}\hat{V}, \tag{6.8}$$

with matrices $\tilde{J}$ and $\tilde{M}$ as in (6.6). Notice that the equation (6.8)-(6.6) is Hamiltonian according to the definition in Section 10.3, and the conservation law (10.52) applies yielding

$$\begin{aligned} \hat{V}^*\tilde{J}\hat{V} &= \mathrm{i}\left[\widehat{\mathsf{p}}_z^*(-\mathrm{i}\omega\widehat{\mathsf{q}}(z)) + (-\mathrm{i}\omega\widehat{\mathsf{q}}(z))^*\,\widehat{\mathsf{p}}_z\right] = 2\mathrm{i}\,\mathrm{Re}\left\{(-\mathrm{i}\omega\widehat{\mathsf{q}}(z))^*\,\widehat{\mathsf{p}}_z\right\} \\ &= -2\mathrm{i}\omega\,\mathrm{Im}\left\{(\widehat{\mathsf{q}}(z))^*\left[\theta(-\mathrm{i}\omega)\,\widehat{\mathsf{q}}(z) - \eta(z)\,\partial_z\widehat{\mathsf{q}}(z)\right]\right\} = \text{constant}. \end{aligned} \tag{6.9}$$

Later on, in Section 8.1, we will see how the above conservation law relates to energy flux constancy.

7 Amplification for the homogeneous case

This subsection is devoted to the analysis of the amplification regime associated with a single exponentially growing mode in the case of an homogeneous MTLB system, that is with parameters not varying with z. For real ω we seek solutions of (4.13) in the form

$$Q(z,t) = \widehat{Q}\mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \qquad q(z,t) = \widehat{q}\mathrm{e}^{-\mathrm{i}(\omega t - kz)}, \tag{7.1}$$

where $\widehat{q}$ and k are complex constants and $\widehat{Q}$ is a complex vector. We show that, under certain conditions, there is a solution with genuinely complex, that is, non real, wave number k.

The symmetric $n \times n$ matrix $L^{-1/2}C^{-1}L^{-1/2}$ is positive definite with eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$, where multiple eigenvalues are repeated according to their multiplicity. Then the MTL has characteristic velocities

$$\pm v_i, \text{ where } v_i^2 = \lambda_i. \tag{7.2}$$

We show below that if either

(i) $0 < u_0 \in (0, v_1]$ or

(ii) $0 < u_0 \notin (0, v_1]$ and $\xi > 0$ is sufficiently small,

then there are *exactly two complex conjugate* values k_0 and k_0^* of $k = k(\omega)$ such that (7.1) is a non-trivial solution of equations (4.13). Hence, assuming $\operatorname{Im} k_0 < 0$ we have the associated solution

$$Q(z,t) = A(z)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-(\operatorname{Im} k_0)z}, \qquad q(z,t) = B(z)\mathrm{e}^{-\mathrm{i}\omega t}\mathrm{e}^{-(\operatorname{Im} k_0)z}, \qquad A(z), B(z) \neq 0, \tag{7.3}$$

that grows exponentially in the $+z$ direction, whereas the solution associated with k_0^* decays exponentially.

We start the analysis by plugging the expressions (7.1) into the system (4.13), obtaining the following linear algebraic system of $n+1$ equations for $\widehat{Q}, \widehat{q}$:

$$\begin{bmatrix} -v^2 L + C^{-1} & D \\ D^T & d - \xi(v-u_0)^2 \end{bmatrix} \begin{bmatrix} \widehat{Q} \\ \widehat{q} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{where } v = \frac{\omega}{k} \tag{7.4}$$

and

$$D = (D_i), \qquad D_i = \sum_j (C^{-1})_{ij}, \qquad d = \sum_i D_i. \tag{7.5}$$

For the sake of brevity, we denote

$$A(v) = -v^2 L + C^{-1}, \qquad \widetilde{A}(v) = \begin{bmatrix} A(v) & D \\ D^T & d - \xi(v-u_0)^2 \end{bmatrix}. \tag{7.6}$$

The system (7.4) has nontrivial solutions if and only if $\left|\widetilde{A}(v)\right| = 0$. The corresponding polynomial equation of degree $2n+2$ is the dispersion relation of our system written in terms of the velocity v. In Section 10.5 we prove in full detail that the equation $\left|\widetilde{A}(v)\right| = 0$ has exactly one pair of complex conjugate solutions if either (i) or (ii) holds. Here we outline the main ideas of the proof.

First of all, the following *canonical factorization* takes place

$$\left|\widetilde{A}(v)\right| = |A(v)| \left[d - \xi(v-u_0)^2 - D^T (A(v))^{-1} D\right], \tag{7.7}$$

(more precisely, such a factorization holds if $|A(v)| \neq 0$, but it also holds in the limit when $|A(v)| = 0$). Therefore, the roots of $\left|\widetilde{A}(v)\right| = 0$ different from $\pm v_i$, $i = 1, 2, ...n$ are the roots of the equation

$$-\xi(v-u_0)^2 = R(v), \text{ where } R(v) = D^T (A(v))^{-1} D - d, \tag{7.8}$$

in which the two components of the system enter separately. The rational function $R(v)$ in (7.8) contains the relevant information about the MTL whereas the left hand side depends only on the beam parameters. In what follows we refer to function $R(v)$ as *MTL characteristic function.* It can be explicitly written in terms of the characteristic velocities:

$$R(v)=\sum_{1}^{n}\frac{\widetilde{D}_i^2}{v_i^2-v^2}-d,$$

where $\widetilde{D}_i$ are constants related to D_i, see Section 10.5. The graph of the MTL characteristic function R symmetric with respect to the vertical axis and is made up of branches, a central one with the minimum at $(0,0)$, a number of increasing branches for $v>0$ and decreasing for $v<0$. One can readily see that $\lim_{v\to\infty}R(v)=-d$. In addition to that, the graph of R has vertical asymptotes at $v=\pm v_i$ if at least one of the associated $\widetilde{D}_i$ does not vanish. The number of the asymptotes varies between 2 and $2n$. The left-hand side in (7.8) is a parabola with vertex at $(u_0,0)$.

Figure 7.1 shows the graph of R and that of the parabola $y=-\xi(v-u_0)^2$ for with the following inductance and capacity matrices

$$L=\begin{bmatrix}4 & 1 & 1/2\\ 1 & 5 & 2\\ 1/2 & 2 & 2\end{bmatrix};\qquad C=\begin{bmatrix}2 & 1 & 2\\ 1 & 4 & 0\\ 2 & 0 & 1\end{bmatrix}.$$

The approximate values of the characteristic velocities are: $v_1=0.18357$ and $v_2=0.42383$. In Figure 7.1 (a), $u_0=0.18$ and $\xi=2$; in Figure 7.1 (b), $u_0=0.8$ and $\xi=18$.

It is important to observe that the parabola always intersects all the branches of R except for the central one. For small ξ each branch is intersected only once, and consequently the number of real roots of the equation (7.8) is exactly the number of asymptotes, as in Figure 7.1 (a) above. For large ξ however the number of real roots can exceed the number of asymptotes as in Figure 7.1 (b), where a large value of ξ produces three points of intersection with the far right branch of the graph of R. Moreover, if $u_0\le v_1$ (geometrically, the vertex of the parabola lies between the vertical axis and the first asymptote), then clearly the number of real roots equals the number of asymptotes irrespective of the value of $\xi>0$. These facts can be proved rigorously based on monotonicity properties, but their geometric interpretation is so transparent that a quick look at Figure 7.1 is quite convincing.

In the generic case, the roots of the dispersion relation $\left|\widetilde{A}(v)\right|=0$ are exactly those of equation (7.8), but in general some of the v_i can also be roots. Whenever some v_i is a real root (maybe multiple) of $\left|\widetilde{A}(v)\right|=0$, the number of asymptotes in the graph of R is reduced by the corresponding amount. The same is true of the number of real roots of (7.8) under either (i) or (ii). This fact follows from factorization (7.7). The main point is that in all cases the total number of real roots of $\left|\widetilde{A}(v)\right|=0$ is $2n$ if either condition (i) or (ii) above holds. We thus conclude that under (i) or (ii) there is necessarily a *unique* pair of complex conjugate roots. The detailed proof of these facts is provided in Section 10.5.

It is not difficult to estimate how small ξ should be in condition (ii) above. In the case $n=1$ and $u_0>v_1$ there is a precise criterion on ξ, namely amplification takes place if

$$\xi<\xi_0:=\frac{L\gamma^2}{1-\gamma^{2/3}};\qquad \gamma=\frac{v_1}{u_0}=\frac{1}{u_0\sqrt{LC}}.\tag{7.9}$$

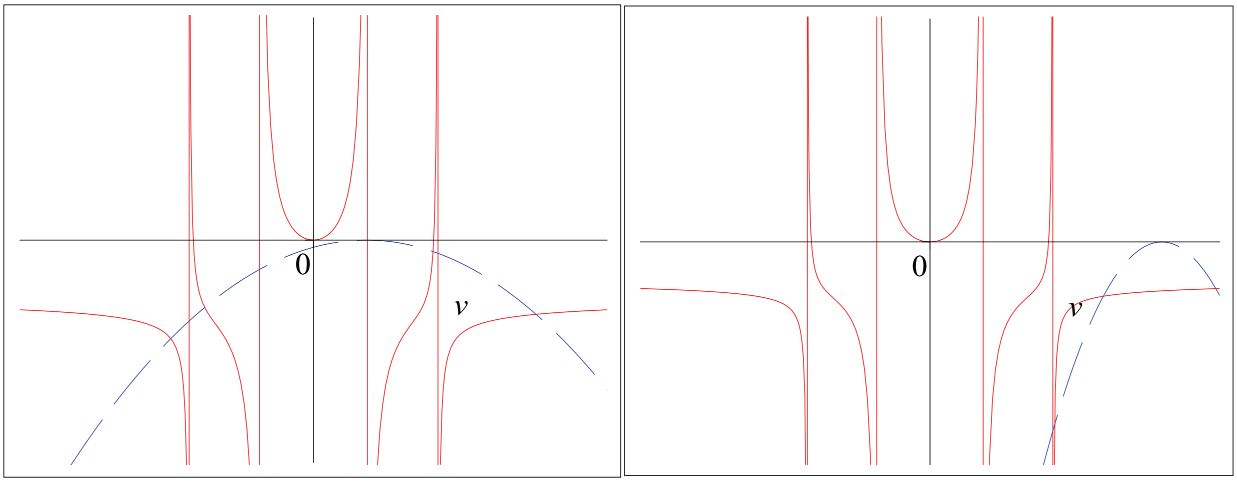

Figure 7.1: (a) $u_0 < v_1$: the parabola (dashed line) intersects each branch of R just once. Four real roots. (b) $u_0 > v_1$: for large ξ, the parabola intersects one of the branches three times. Six real roots.

A simple sufficient condition can be also given for $n > 1$. For example, one can just impose that the left branch of the parabola at $v = 0$ be flatter than the flattest point of the graph of R on (v_1, u_0).This leads to

$$\xi < \widetilde{\xi}_0 := \frac{\min_{v\in(v_1,u_0)} R'(v)}{2u_0}. \tag{7.10}$$

Observe that both ξ_0 and $\widetilde{\xi}_0$ vanish as $u_0 \to \infty$, as expected. The value of $\widetilde{\xi}_0$ is not sharp but we did not make an effort to find one.

7.1 Asymptotic behavior of the amplification factor as $\xi \to 0$ and as $\xi \to \infty$.

Let k_0 denote the complex root with $\operatorname{Im} k_0 < 0$ whose existence we proved in the previous section under appropriate conditions. It is interesting to study the asymptotics of the "amplification factor" $-\operatorname{Im} k_0$ as the beam parameter $\xi \to 0$, as well as its behavior when $\xi \to \infty$. A careful analysis shows (see Section 10.5) that, if we denote by $v_0 = \omega/k_0$ the corresponding velocity with $\operatorname{Im} v_0 > 0$, then

$$\operatorname{Im} v_0 = \sqrt{K'\xi + o(\xi)} = \sqrt{K'}\sqrt{\xi} + o(\sqrt{\xi}) \quad \text{as } \xi \to 0$$

where K' depends only on L, C, u_0. As a consequence,

$$-\operatorname{Im} k_0 = \frac{\operatorname{Im} v_0}{|v_0|^2} \sim \frac{K''}{\sqrt{\xi}} \quad \text{as} \quad \xi \to 0; \quad K'' > 0. \tag{7.11}$$

The conclusion is that, at least in theory, the amplification factor can be indefinitely improved by reducing ξ. This amounts to increasing ρ_0, the electron density of the beam.

On the other hand, the limit $\xi \to \infty$ makes sense only if $0 < u_0 \le v_1$. In the case of one line and $u_0 = v_1$, it can be proved that

$$-\operatorname{Im} k_0 = \frac{\operatorname{Im} v_0}{|v_0|^2} \sim \frac{K'''}{\sqrt[3]{\xi}} \quad \text{as} \quad \xi \to \infty; \quad K''' > 0, \tag{7.12}$$

see Section 10.5.

The regime considered by Pierce corresponds to the latter situation, in which there are two real solutions (for v) close to $\pm u_{0,}$ and two complex conjugate with real part close to u_0; see Section 9. The situation is similar for $u_0 < v_1$, but in this case $-\operatorname{Im} k_0$ has a finite positive limit as $\xi \to \infty$.

8 Energy conservation and transfer

The conservation laws for our system can be obtained via Noether theorem, [GelFom, 38.2-3], [Gold, 13.7]. First of all, the total energy of the system is conserved. This is a consequence of the closedness of the system, reflected as explicit independence of the Lagrangian density $\mathcal{L}$ on t. Namely, the energy and the energy flux densities are respectively

$$H = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_t \mathsf{q}_j)} \partial_t \mathsf{q}_j - \mathcal{L}, \quad S = \sum_j \frac{\partial \mathcal{L}}{\partial(\partial_z \mathsf{q}_j)} \partial_t \mathsf{q}_j, \tag{8.1}$$

and the following *energy conservation law* holds

$$\partial_t H + \partial_z S = -\partial_t \mathcal{L} = 0. \tag{8.2}$$

In particular, for a system governed by the quadratic Lagrangian density (10.13) we have

$$\text{total energy density: } H = \frac{1}{2} \partial_t \mathsf{q}^{\mathrm{T}} \alpha \partial_t \mathsf{q} + \frac{1}{2} \partial_z \mathsf{q}^{\mathrm{T}} \eta \partial_z \mathsf{q}, \tag{8.3}$$

$$\text{total energy flux: } S = \partial_t \mathsf{q}^{\mathrm{T}} \theta \partial_t \mathsf{q} - \partial_t \mathsf{q}^{\mathrm{T}} \eta \partial_z \mathsf{q} = \partial_t \mathsf{q}^{\mathrm{T}} \left(\theta \partial_t \mathsf{q} - \eta \partial_z \mathsf{q} \right) = \partial_t \mathsf{q}^{\mathrm{T}} \mathsf{p}_z, \tag{8.4}$$

where p_z is the canonical momentum defined in (10.17).

Consider now a real time harmonic eigenmode

$$\mathsf{q}(t, z) = \operatorname{Re}\left\{ \hat{\mathsf{q}}(z) \mathrm{e}^{-\mathrm{i}\omega t} \right\}, \text{ with a complex valued } \hat{\mathsf{q}}(z), \tag{8.5}$$

which solves the Euler-Lagrange equation (10.16). Notice that $\langle \mathsf{q} \rangle (z) = 0$, where $\langle \cdot \rangle$ is the time average operation defined in (5.9). However, if

$$a(t) = \operatorname{Re}\left\{ \hat{a} \mathrm{e}^{-\mathrm{i}\omega t} \right\}, \text{ with a complex valued } \hat{a} \tag{8.6}$$

and $b(t)$ is defined by a similar formula then we have

$$\langle ab \rangle = \frac{1}{2} \operatorname{Re}\left\{ \hat{a}^* \hat{b} \right\}. \tag{8.7}$$

Applying the average operation $\langle \cdot \rangle$ to the conservation law (8.2) for a time harmonic eigenmode q as in (8.5) and using (5.10) and (5.11), we obtain

$$\partial_z \langle S \rangle (z) = 0 \text{ implying } \langle S \rangle (z) = \text{constant}. \tag{8.8}$$

On the other hand, S defined by (8.4) can be written as the product of two real time harmonic functions:

$$S(t, z) = \operatorname{Re}\left\{ \widehat{A}(z) \mathrm{e}^{-\mathrm{i}\omega t} \right\} \operatorname{Re}\left\{ \widehat{B}(z) \mathrm{e}^{-\mathrm{i}\omega t} \right\}, \tag{8.9}$$

where

$$\widehat{A}(z) = -i\omega\hat{\mathsf{q}}\,(z)\,; \qquad \widehat{B}(z) = -i\omega\theta\hat{\mathsf{q}}(z) - \eta\partial_z\hat{\mathsf{q}}(z). \tag{8.10}$$

Using (8.7) we obtain the energy flux conservation law in the form

$$\langle S\rangle\,(z) = \frac{1}{2}\,\mathrm{Re}\left\{\langle(-i\omega\hat{\mathsf{q}})^*\,(-i\omega\theta\hat{\mathsf{q}} - \eta\partial_z\hat{\mathsf{q}})\rangle\right\} = \frac{1}{2}\,\mathrm{Re}\left\{\langle(-i\omega\hat{\mathsf{q}})^*\,\hat{\mathsf{p}}_z\rangle\right\} = \text{constant}\,. \tag{8.11}$$

Constancy of $\langle S\rangle\,(z)$ is related to the constancy of the symplectic square of the solution of the Hamiltonian system satisfied by

$$\widehat{V}(z) = \begin{bmatrix} \widehat{\mathsf{p}}_z \\ -i\omega\widehat{\mathsf{q}} \end{bmatrix}, \tag{8.12}$$

see formula (6.9). Indeed,

$$V^*\tilde{J}V = 2i\,\mathrm{Re}\left\{(-i\omega\hat{\mathsf{q}}\,(z))^*\,[-\mathrm{i}\omega\theta\hat{\mathsf{q}}\,(z) - \eta\partial_z\hat{\mathsf{q}}\,(z)]\right\} = \text{constant} = 4\mathrm{i}\,\langle S\rangle\,(z)\,. \tag{8.13}$$

8.1 Energy exchange between subsystems

This section deals with the balance of energy between the two subsystems making up our system: the beam and the MTL. As already pointed out by Pierce in [Pier51, p. 635] an amplification regime assumes that the energy extracted from the beam is stored in the EM field. In other words, the net flux of energy must have a definite sign. Pierce tacitly considers this condition as an additional one to be imposed on top of other conditions ensuring the existence of an exponentially growing solution. We show here that in fact this condition is automatically satisfied for exponentially growing solutions.

When computing the energy flux between the beam and the MTL we take advantage of our Lagrangian setting. This setting allows for a systematic derivation of expressions for energies and fluxes satisfying *a priori* the fundamental conservation laws. We proceed using the results from Section 10.4 for a more general coupled system.

First, we should split the Lagrangian into two parts $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ corresponding to the MTL and the beam. Namely,

$$\mathcal{L}_1(Q_t, Q_{;z}) = \frac{1}{2}\left(\partial_t Q, L\partial_t Q\right)^2 - \frac{1}{2}\left(\partial_{;z}Q, C^{-1}\partial_{;z}Q\right)^2; \tag{8.14}$$

$$\mathcal{L}_2(q_t, q_z) = \frac{\xi}{2}(\partial_t q + u_0\partial_z q)^2, \text{ where } \partial_{;z}Q = \partial_z Q + B\partial_z q.$$

The above Lagrangian has the structure of (10.53), with $B = (1, 1...1)^{\mathrm{T}}$. According to (10.62), the power $P_{\mathrm{B}\to\mathrm{MTL}}$ flowing from the beam to the (unit length of) the MTL is given by

$$P_{\mathrm{B}\to\mathrm{MTL}} = -\frac{\partial\mathcal{L}_1}{\partial(\partial_{;z}Q)}\partial^2_{tz}q = \partial_{;z}Q^T C^{-1} B\partial^2_{tz}q = \partial_z I_b \sum_i D_i\partial_{;z}Q_i, \tag{8.15}$$

$$\text{where } D_i = \sum_j (C^{-1})_{ij}.$$

Using (4.14) we recast the expression for $P_{\mathrm{B}\to\mathrm{MTL}}$ in terms of currents and voltages. Indeed we define the voltage V by

$$V = -C^{-1}(\partial_z Q + \partial_z q). \tag{8.16}$$

Then we notice that

$$\sum_i D_i \partial_{;z} Q_i = \sum_i \sum_j (C^{-1})_{ij} (\partial_z Q_i + \partial_z q B_i) = -\sum_j V_j$$

and hence, according to (4.14),

$$P_{\mathrm{B}\to\mathrm{MTL}} = -\sum_j \partial_z I_b V_j = -\left(\partial_z I_b B, V\right) = (C\partial_t V, V) + (\partial_z I, V) = \tag{8.17}$$

$$= \partial_t \left[\frac{1}{2}\left(CV, V\right)\right] + \partial_z \left(I, V\right) - \left(I, \partial_z V\right) = \partial_t \left[\frac{1}{2}\left(CV, V\right)\right] + \left(L\partial_t I, I\right) + \partial_z \left(I, V\right) =$$

$$= \partial_t \left[\frac{1}{2}\left(CV, V\right) + \frac{1}{2}\left(LI, I\right)\right] + \partial_z (I, V).$$

where, as usual, $(,)$ stands for the scalar product. The first two terms above correspond to $\partial_t H$ where

$$H = \frac{1}{2}(CV, V) + \frac{1}{2}(LI, I) \tag{8.18}$$

is the density of the total energy stored in the shunt capacitors and the inductances per unit length. The last term in $P_{\mathrm{B}\to\mathrm{MTL}}$ represents the divergence of the energy flux, $S = (I, V)$. In the particular case of one line, we recover the usual expressions for the corresponding quantities:

$$P_{\mathrm{B}\to\mathrm{MTL}} = \partial_t \left[\frac{1}{2}CV^2\right] + \partial_t \left[\frac{1}{2}LI^2\right] + \partial_z (IV). \tag{8.19}$$

Next, we prove the following fundamental property of general homogeneous MTLB systems. *Under the assumptions of existence of an exponentially growing wave, the energy on such growing solution always flows from the beam to the MTL.* To see that we observe first that for real time harmonic solutions Q and q of the form

$$Q = \mathrm{Re}\left(\widehat{Q}\mathrm{e}^{i(kz-\omega t)}\right), \qquad q = \mathrm{Re}\left(\widehat{q}\mathrm{e}^{i(kz-\omega t)}\right), \tag{8.20}$$

where $\widehat{Q}, \widehat{q}$ are complex constants, the expression for $P_{\mathrm{B}\to\mathrm{MTL}}$ can be written in the form

$$P_{\mathrm{B}\to\mathrm{MTL}} = \partial_{;z} Q^T C^{-1} B \partial_{tz}^2 q = \mathrm{Re}(\widehat{a}(z)e^{-i\omega t})\,\mathrm{Re}(\widehat{b}(z)e^{-i\omega t}), \tag{8.21}$$

where

$$\widehat{a}(z) = ike^{ikz}(\widehat{Q} + B\widehat{q})^T C^{-1}; \qquad \widehat{b}(z) = -\omega k \widehat{q} e^{ikz} B. \tag{8.22}$$

Applying formula (8.7) for time average, we get

$$\left\langle P_{\mathrm{B}\to\mathrm{MTL}}\right\rangle(z) = -\frac{\omega}{2}\mathrm{e}^{-2(\mathrm{Im}\,k)z}\,\mathrm{Im}\left\{|k|^2 \left(\widehat{Q} + B\widehat{q}\right)^{*T} C^{-1} B\widehat{q}\right\}. \tag{8.23}$$

Suppose now that k_0 is the complex root providing amplification, that is, in the notation of Subsection 7.1, $k_0 = \omega/v_0$ with $\mathrm{Im}\,k_0 < 0$. Then, v_0 is a root of the system (7.4) and therefore, in the notation of Section 7 and returning to the variable k,

$$k_0^2(\widehat{Q}^{\mathrm{T}} D + \widehat{q} d) = \xi(\omega - k_0 u_0)^2 \widehat{q}. \tag{8.24}$$

Observing that $C^{-1}B = D$ and $BC^{-1}B = d$,we can rewrite (8.23) in the form

$$\langle P_{\mathrm{B}\to\mathrm{MTL}}\rangle (z) = -\frac{\omega\xi}{2}\mathrm{e}^{-2(\mathrm{Im}\,k_0)z}\,\mathrm{Im}\left\{\frac{|k_0|^2}{k_0^2}(\omega - u_0k_0)^{*2}\,|\widehat{q}|^2\right\} \tag{8.25}$$

$$= -\frac{\omega\xi\,|k_0|^2\,|\widehat{q}|^2\,u_0^2}{2}\mathrm{e}^{-2(\mathrm{Im}\,k_0)z}\,\mathrm{Im}\left\{\left(\frac{k_b - k_0^*}{k_0^*}\right)^{*2}\right\}, \qquad k_b = \frac{\omega}{u_0}.$$

In terms of velocities, we have

$$\mathrm{Im}\left(\frac{k_b - k_0^*}{k_0^*}\right)^{*2} = \mathrm{Im}\left(\frac{v_0}{u_0} - 1\right)^2 = \frac{1}{u_0^2}(\mathrm{Re}\,v_0 - u_0)\,\mathrm{Im}\,v_0. \tag{8.26}$$

Since we are assuming $\mathrm{Im}\,v_0 > 0$, we see from formula (8.25) that $\langle P_{\mathrm{B}\to\mathrm{MTL}}\rangle (z) \geq 0$ for all z exactly if $\mathrm{Re}\,v_0 \leq u_0$. But this is always the case, as it follows from (10.70) and (10.72).

Observe now that

$$\mathrm{Im}\,k_0 = -\,\mathrm{Im}\,v_0/\,|v_0|^2 < 0 \tag{8.27}$$

and formula (8.25) implies that $\langle P_{\mathrm{B}\to\mathrm{MTL}}\rangle$ increases in the $+z$ direction. For the evanescent wave, corresponding to the value k_0^*, we have exactly the opposite situation: the energy flows from the MTL to the beam and the power flux decreases in the $+z$ direction.

9 Pierce model revisited

Let us examine Pierce's original results in the light of our general theory. They correspond to $n = 1$, hence $d = D = C^{-1}$ and the dispersion relation $\left|\widetilde{A}(v)\right| = 0$ becomes

$$\left(-v^2L + C^{-1}\right)\left[C^{-1} - \xi(v - u_0)^2\right] - C^{-2} = 0, \tag{9.1}$$

which, in terms of $k = \omega/v$, reads

$$-\,L\omega^2k^2 + \xi(\omega - ku_0)^2(LC\omega^2 - k^2) = 0. \tag{9.2}$$

After elementary algebraic transformations the above equation turns into

$$u_0^2k^4 - 2u_0\omega k^3 + \left[1 + \frac{L}{\xi} - LCu_0^2\right]\omega^2k^2 + 2LCu_0\omega^3k - LC\omega^4 = 0, \tag{9.3}$$

which is precisely the fourth order equation in [Pier51, (1.16)]

The TL has only two characteristic velocities, namely $\pm v_1 = \pm 1/\sqrt{LC}$ which are not solutions of (9.1). The graph of the characteristic function R has only two vertical asymptotes at $v = \pm v_1$. The special regime considered in [Pier51] corresponds to taking $u_0 \leq v_1$ and large ξ, and the author focuses on the particular case $u_0 = v_1$. As we know, in this case amplification occurs for any $\xi > 0$. For small values of the parameter

$$k_p = \frac{\omega_p}{u_0} = \frac{1}{u_0}\sqrt{\frac{4\pi}{\sigma\xi}}, \tag{9.4}$$

Pierce asserts that $k \simeq k_{\mathrm{b}} = \omega/u_0$ for the forward unattenuated wave. In terms of velocities this means that for large values of ξ the positive real solution v_1^+ is very close to u_0. The

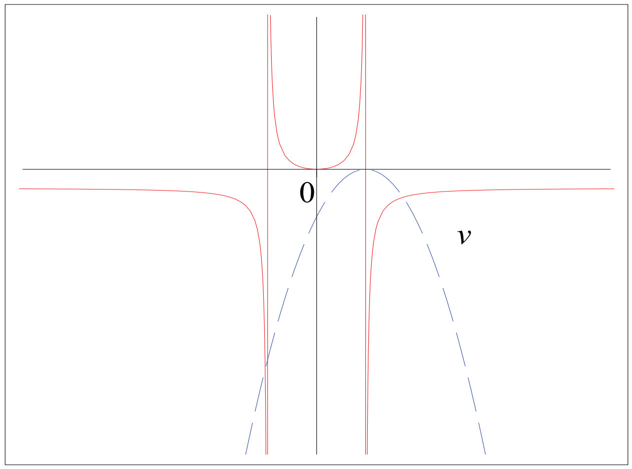

Figure 9.1: Pierce dispersion relation: For large ξ, the parabola intersects the graph of R close to the asymptotes: $v_1^+, v_1^- \approx u_0$.

graph in Figure 9.1 refers to this situation and it clearly shows that indeed $v_1^+ \simeq u_0$ and $-v_1^- \simeq -u_0$ for large ξ (the parabola becomes very narrow and the right and left branches of the graph of R are intersected close to the asymptotes).

Consequently, the identity

$$2\,\mathrm{Re}\,v_0 + v_1^+ + v_1^- = 2u_0 \tag{9.5}$$

implies that $\mathrm{Re}\,v_0$, $\mathrm{Re}\,v_0^* \simeq u_0$. In order to get rid of the backward wave (corresponding to $-v_1^-$) Pierce looks for solutions (in k) to the fourth order equation (9.3) in the form

$$k = k_\mathrm{b} + \mathrm{i}\delta,$$

with the complex number δ small in magnitude compared to k_b. The dispersion relation (9.2) then reads

$$(\mathrm{i}\delta)^3\left(2 + \mathrm{i}\delta k_\mathrm{b}^{-1}\right) = -L\xi^{-1}k_\mathrm{b}^2\left(1 + \mathrm{i}\delta k_\mathrm{b}^{-1}\right)^2. \tag{9.6}$$

Neglecting $\mathrm{i}\delta/k_\mathrm{b}$ we arrive at Pierce's third degree equation for δ:

$$\delta^3 = -\frac{Lk_\mathrm{b}^2\xi^{-1}}{2}\mathrm{i}, \tag{9.7}$$

which has three complex roots,

$$\delta_1 = c\mathrm{i}, \qquad \delta_2 = c\left(-\sqrt{3} - \mathrm{i}\right)/2, \qquad \delta_3 = c\left(\sqrt{3} - \mathrm{i}\right)/2, \text{ where } c = \sqrt[3]{Lk_\mathrm{b}^2\xi^{-1}/2}, \tag{9.8}$$

corresponding respectively to the unattenuated wave faster than the natural phase velocity of the circuit ($v_1^+ > v_1 = u_0$), the increasing and the decreasing waves.

It is clear from the analysis in Section 10.5 that in case of several identical, non-interacting TLs, only two asymptotes are present in the graph of R. This might lead to think that we can replace such a system by a single effective line, with modified parameters, interacting with the beam. But in fact this is not the case. Indeed, let $C = \widehat{C}\mathrm{Id}_n$, $L = \widehat{L}\mathrm{Id}_n$ with $n \geq 2$. Then, there are exactly two characteristic velocities $\pm v_n$ where $v_n = 1/\sqrt{\widehat{L}\widehat{C}}$. According to Section 10.5 these are necessarily characteristic velocities of the entire system, of multiplicity

$n-1$ each. Using the notation from that section, we have $D = \widehat{C}^{-1}(1,1,...1)^T$, $d = n\widehat{C}^{-1}$ and $A(v)$ is already diagonal. The MTL characteristic function $R(v)$ has the explicit expression

$$R(v) = \frac{n\widehat{C}^{-2}}{v_n^2 - v^2} - n\widehat{C}^{-1}, \tag{9.9}$$

and, consequently, the dispersion relation takes the form

$$-\xi(v-u_0)^2 = \frac{n\widehat{C}^{-2}}{v_n^2 - v^2} - n\widehat{C}^{-1}. \tag{9.10}$$

It is evident that this relation cannot be reduced to the dispersion relation for just one line with the same beam. It follows after dividing by n that the interaction of this system with a beam with parameters (ξ, u_0) is equivalent to the interaction of one line with parameters $\widehat{L}$, $\widehat{C}$ with a beam with parameters $(\xi/n, u_0)$. The asymptotic formula (7.11) then implies that the amplification factor grows as $\sqrt{n}$ as $n \to \infty$.

10 Mathematical subjects

In this section we consider subjects related to our studies that are mathematically involved.

10.1 de Donder-Weyl version of the Hamiltonian formalism

In this section we introduce basic settings of the de Donder-Weyl (DW) version of the Hamilton equations which treats the time and space variable in equal manner just as the Lagrangian approach which constitutes its basis. The DW theory is a generalization of the standard Hamiltonian formalism and the Hamilton-Jacobi theory, [Rund, 4.2] that has the advantage of requiring a finite-dimensional phase space. We do not use any significant results of the DW theory but rather take advantage of its set up that allows to treat the time t and the space variable z on equal footing. We remind that the standard Hamilton-Jacobi theory gives preferential treatment to time t.

Let us consider a system $\mathsf{q} = \{\mathsf{q}_j(t,z),\ j = 1,\dots n\}$ of real valued fields depending on time t and one-dimensional space variable z. Suppose it has a Lagrangian density of the form

$$\mathcal{L} = \mathcal{L}(t, z, \mathsf{q}, \mathsf{q}_{,t}, \mathsf{q}_{,z}),\ \text{where } \mathsf{q}_{,t} = \partial_t \mathsf{q},\ \mathsf{q}_{,z} = \partial_z \mathsf{q}. \tag{10.1}$$

The corresponding Euler-Lagrange equations are, [GelFom, 4.16]

$$\frac{\partial \mathcal{L}}{\partial \mathsf{q}} - \partial_t \frac{\partial \mathcal{L}}{\partial \mathsf{q}_{,t}} - \partial_z \frac{\partial \mathcal{L}}{\partial \mathsf{q}_{,z}} = 0. \tag{10.2}$$

Evidently, (10.2) is a system of second order partial differential equations for q as a function of t, z. It can be recast as the first order partial differential system with respect to time t or with respect to the space variable z using a generalization of the standard Hamiltonian formalism known as de Donder-Weyl (DW) theory. Thus, following the DW theory we introduce two canonical momenta densities p_t and p_z and the DW Hamiltonian density $\mathcal{H}$ by the formulas

$$\mathsf{p}_t = \frac{\partial \mathcal{L}}{\partial \mathsf{q}_{,t}}(t, z, \mathsf{q}, \mathsf{q}_{,t}, \mathsf{q}_{,z}), \tag{10.3}$$

$$\mathsf{p}_z = \frac{\partial \mathcal{L}}{\partial \mathsf{q}_{,z}}(t, z, \mathsf{q}, \mathsf{q}_{,t}, \mathsf{q}_{,z}), \tag{10.4}$$

$$\mathcal{H}_{\mathrm{DW}} = \mathcal{H}_{\mathrm{DW}}(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z) = \mathsf{p}_t^{\mathrm{T}} \mathsf{q}_{,t} + \mathsf{p}_z^{\mathrm{T}} \mathsf{q}_{,z} - \mathcal{L}(t, z, \mathsf{q}, \mathsf{q}_{,t}, \mathsf{q}_{,z}), \tag{10.5}$$

where $\mathsf{q}_{,t}$ and $\mathsf{q}_{,z}$ are supposed to be found from respective equations (10.3)-(10.4) and to be substituted in the right-hand side for the second equation in (10.5). Then the corresponding DW version of the Hamilton equations are

$$\partial_t \mathsf{q} = \frac{\partial \mathcal{H}_{\mathrm{DW}}}{\partial \mathsf{p}_t}\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right), \tag{10.6}$$

$$\partial_z \mathsf{q} = \frac{\partial \mathcal{H}_{\mathrm{DW}}}{\partial \mathsf{p}_z}\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right), \tag{10.7}$$

$$\partial_t \mathsf{p}_t + \partial_z \mathsf{p}_z = -\frac{\partial \mathcal{H}_{\mathrm{DW}}}{\partial \mathsf{q}}\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right), \tag{10.8}$$

and this system of $3n$ first order equations is equivalent to the Euler-Lagrange system (10.2).

One can solve the system (10.3) -(10.4) for $\mathsf{q}_{,t}$ and $\mathsf{q}_{,z}$ in terms of the momenta, obtaining representations

$$\mathsf{q}_{,t} = G_t\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right), \qquad \mathsf{q}_{,z} = G_z\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right), \tag{10.9}$$

for some functions G_t and G_z. Solving for p_t in the first and for p_z in the second, we get

$$\mathsf{p}_t = K_t\left(t, z, \mathsf{q}, \mathsf{q}_{,t}, \mathsf{p}_z\right), \qquad \mathsf{p}_z = K_z\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{q}_{,z}\right), \tag{10.10}$$

for some functions K_t and K_z.

To obtain the first order partial differential equations with respect to t we consider the pair p_t, q and using equations (10.6) and (10.8) we get

$$\begin{aligned} \partial_t \mathsf{q} &= \frac{\partial H_{\mathrm{DW}}}{\partial \mathsf{p}_t}\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right) = F_{\mathsf{q}}\left(t, z, \mathsf{q}, \mathsf{q}_{,z}, \mathsf{p}_t\right), \\ \partial_t \mathsf{p}_t &= -\partial_z \mathsf{p}_z - \frac{\partial H_{\mathrm{DW}}}{\partial \mathsf{q}}\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right) = F_{\mathsf{p}}\left(t, z, \mathsf{q}, \mathsf{q}_{,z}, \mathsf{q}_{,zz}, \mathsf{p}_t, \mathsf{p}_{t,z}\right), \end{aligned} \tag{10.11}$$

where the expressions F_{q} and F_{p} are obtained by plugging the representation (10.10) for p_z into the relevant expressions in (10.11). Observe that the system of partial differential equations (10.11) for p_t and q is of the first order with respect to time t.

To obtain the first order partial differential equation with respect to z we consider the pair p_z, q and proceed just as in the previous case with using the equations (10.7) and (10.8) to get

$$\begin{aligned} \partial_z \mathsf{q} &= \frac{\partial H_{\mathrm{DW}}}{\partial \mathsf{p}_z}\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right) = F_{\mathsf{q}}\left(t, z, \mathsf{q}, \mathsf{q}_{,t}, \mathsf{p}_z\right), \\ \partial_z \mathsf{p}_z &= -\partial_t \mathsf{p}_t - \frac{\partial H_{\mathrm{DW}}}{\partial \mathsf{q}}\left(t, z, \mathsf{q}, \mathsf{p}_t, \mathsf{p}_z\right) = F_{\mathsf{p}}\left(t, z, \mathsf{q}, \mathsf{q}_{,t}, \mathsf{q}_{,tt}, \mathsf{p}_z, \mathsf{p}_{z,t}\right), \end{aligned} \tag{10.12}$$

where the expressions F_{q} and F_{p} are determined by plugging the representation (10.10) for p_t into the relevant expressions in (10.12). Observe that the system of partial differential equations (10.12) for p_z and q is of the first order with respect to the space variable z.

10.2 Quadratic Lagrangian densities

In this section we present some results concerning a special family of Lagrangians, namely those quadratic in the derivatives (and independent both of coordinates and the fields). This

kind of Lagrangians often appear in practice, in particular in the TL-beam interaction system. Thus, let us consider a quadratic Lagrangian density of the form

$$\mathcal{L}(\mathsf{q}_{,t},\mathsf{q}_{,z})=\frac{1}{2}\partial_t\mathsf{q}^{\mathrm{T}}\alpha\partial_t\mathsf{q}+\partial_t\mathsf{q}^{\mathrm{T}}\theta\partial_z\mathsf{q}-\frac{1}{2}\partial_z\mathsf{q}^{\mathrm{T}}\eta\partial_z\mathsf{q},\tag{10.13}$$

where $\mathsf{q} = \{\mathsf{q}_j(t,z),\ j=1,\dots n\}$ are real valued fields depending on time t and one-dimensional space variable z, $\mathsf{q}_{,t}=\partial_t\mathsf{q}$, $\mathsf{q}_{,z}=\partial_z\mathsf{q}$ and $\alpha(t,z)$, $\eta(t,z)$, $\theta(t,z)$ are symmetric $n\times n$ matrices with real entries, that is

$$\alpha^{\mathrm{T}}=\alpha,\qquad \eta^{\mathrm{T}}=\eta,\qquad \theta^{\mathrm{T}}=\theta.\tag{10.14}$$

The Lagrangian density (10.13) can be recast into the following form, involving a block matrix:

$$\mathcal{L}=\frac{1}{2}\mathsf{u}^{\mathrm{T}}M_{\mathrm{L}}\mathsf{u};\qquad M_{\mathrm{L}}=\left[\begin{array}{cc}\alpha & \theta\\ \theta & -\eta\end{array}\right],\qquad \mathsf{u}=\left[\begin{array}{c}\partial_t\mathsf{q}\\ \partial_z\mathsf{q}\end{array}\right].\tag{10.15}$$

The Euler-Lagrange equation (10.2) for this Lagrangian is

$$\left[\partial_t\alpha\partial_t+\partial_t\theta\partial_z+\partial_z\theta\partial_t-\partial_z\eta\partial_z\right]\mathsf{q}=0.\tag{10.16}$$

Now we would like to use the DW Hamiltonian approach from the previous section to recast the second order differential $n\times n$ system (10.16) into first order ones with respect to t and with respect to z as well. With that in mind we introduce the canonical momenta as in (10.3)-(10.4)

$$\mathsf{p}_t=\frac{\partial\mathcal{L}}{\partial\mathsf{q}_{,t}}=\alpha\partial_t\mathsf{q}+\theta\partial_z\mathsf{q},\qquad \mathsf{p}_z=\frac{\partial\mathcal{L}}{\partial\mathsf{q}_{,z}}=\theta\partial_t\mathsf{q}-\eta\partial_z\mathsf{q},\tag{10.17}$$

which can be recast as

$$\mathsf{p}=\left[\begin{array}{c}\mathsf{p}_t\\ \mathsf{p}_z\end{array}\right]=\left[\begin{array}{cc}\alpha & \theta\\ \theta & -\eta\end{array}\right]\left[\begin{array}{c}\partial_t\mathsf{q}\\ \partial_z\mathsf{q}\end{array}\right]=M_{\mathrm{L}}\mathsf{u},\tag{10.18}$$

or

$$\left[\begin{array}{c}\partial_t\mathsf{q}\\ \partial_z\mathsf{q}\end{array}\right]=\mathsf{u}=M_{\mathrm{L}}^{-1}\mathsf{p}=M_{\mathrm{L}}^{-1}\left[\begin{array}{c}\mathsf{p}_t\\ \mathsf{p}_z\end{array}\right].\tag{10.19}$$

Notice that the difference in signs in expressions for momenta p_t and p_z in (10.17) is due to difference in signs for matrices α and η as they enter the expressions for the kinetic and potential energies in the Lagrangian density defined by (10.13).

Solving equations (10.18) for $\partial_t\mathsf{q}$ and $\partial_z\mathsf{q}$ we obtain

$$\partial_t\mathsf{q}=\alpha^{-1}\left(\mathsf{p}_t-\theta\partial_z\mathsf{q}\right),\qquad \partial_z\mathsf{q}=\eta^{-1}\left(\theta\partial_t\mathsf{q}-\mathsf{p}_z\right).\tag{10.20}$$

Using (10.13) and (10.17) we get the following identity

$$\begin{gathered}\mathsf{p}_t^{\mathrm{T}}\partial_t\mathsf{q}+\mathsf{p}_z^{\mathrm{T}}\partial_z\mathsf{q}=\partial_t\mathsf{q}^{\mathrm{T}}\mathsf{p}_t+\partial_z\mathsf{q}^{\mathrm{T}}\mathsf{p}_z=\\ =\partial_t\mathsf{q}^{\mathrm{T}}\left(\alpha\partial_t\mathsf{q}+\theta\partial_z\mathsf{q}\right)+\partial_z\mathsf{q}^{\mathrm{T}}\left(\theta\partial_t\mathsf{q}-\eta\partial_z\mathsf{q}\right)=2L.\end{gathered}\tag{10.21}$$

Then in view of (10.21) the general DW Hamiltonian $\mathcal{H}_{\mathrm{DW}}$ defined by (10.5) takes here the form

$$\mathcal{H}_{\mathrm{DW}}=\mathsf{p}_t^{\mathrm{T}}\partial_t\mathsf{q}+\mathsf{p}_z^{\mathrm{T}}\partial_z\mathsf{q}-\mathcal{L}=\mathcal{L}=\frac{1}{2}\partial_t\mathsf{q}^{\mathrm{T}}\alpha\partial_t\mathsf{q}+\partial_t\mathsf{q}^{\mathrm{T}}\theta\partial_z\mathsf{q}-\frac{1}{2}\partial_z\mathsf{q}^{\mathrm{T}}\eta\partial_z\mathsf{q}.\tag{10.22}$$

Another way to obtain a representation for the DW Hamiltonian is to use (10.19) yielding

$$\mathcal{H}_{\mathrm{DW}}=\mathsf{p}^{\mathrm{T}}\mathsf{u}-\frac{1}{2}\mathsf{u}^{\mathrm{T}}M_{\mathrm{L}}\mathsf{u}=\mathsf{u}^{\mathrm{T}}M_{\mathrm{L}}\mathsf{u}-\frac{1}{2}\mathsf{u}^{\mathrm{T}}M_{\mathrm{L}}\mathsf{u}=\frac{1}{2}\mathsf{u}^{\mathrm{T}}M_{\mathrm{L}}\mathsf{u}=\mathcal{L}=\frac{1}{2}\mathsf{p}^{\mathrm{T}}M_{\mathrm{L}}^{-1}\mathsf{p}. \tag{10.23}$$

Observe based on (10.22) and (10.23) that the DW Hamiltonian $\mathcal{H}$ equals the Lagrangian $\mathcal{L}$ at the corresponding point, that is

$$\mathcal{H}_{\mathrm{DW}}=\mathcal{H}_{\mathrm{DW}}(\mathsf{p})=\mathcal{L}(\mathsf{u})=\mathcal{L},\ \text{ where } \mathsf{p}=M_{\mathrm{L}}\mathsf{u}. \tag{10.24}$$

(actually, this is a general property of the Legendre transform of homogeneous quadratic polynomials). The equation (10.8) takes here the form

$$\partial_t\mathsf{p}_t+\partial_z\mathsf{p}_z=0. \tag{10.25}$$

To obtain the first order equations with respect to t we pick the pair p_t,q. We use equations (10.25) and (10.20) for respectively $\partial_t\mathsf{p}_t$ and $\partial_t\mathsf{q}$. We eliminate p_z in (10.25) by using its representation (10.17) getting the system

$$\partial_t\mathsf{p}_t=-\partial_z\mathsf{p}_z=-\partial_z\theta\partial_t\mathsf{q}+\partial_z\eta\partial_z\mathsf{q}=-\partial_z\theta\alpha^{-1}(\mathsf{p}_t-\theta\partial_z\mathsf{q})+\partial_z\eta\partial_z\mathsf{q}, \tag{10.26}$$

$$\partial_t\mathsf{q}=\alpha^{-1}(\mathsf{p}_t-\theta\partial_z\mathsf{q}). \tag{10.27}$$

Observe that we used equation (10.27) to get the right-hand side of equation (10.26). The above system can be written in matrix form

$$\partial_t\begin{bmatrix}\mathsf{p}_t\\ \mathsf{q}\end{bmatrix}=\begin{bmatrix}-\partial_z\theta\alpha^{-1} & \partial_z\eta\partial_z+\partial_z\theta\alpha^{-1}\theta\partial_z\\ \alpha^{-1} & -\alpha^{-1}\theta\partial_z\end{bmatrix}\begin{bmatrix}\mathsf{p}_t\\ \mathsf{q}\end{bmatrix}. \tag{10.28}$$

One can recast the above system into a canonical Hamiltonian form by using the following symplectic matrix

$$J=\begin{bmatrix}0 & -\mathbf{1}\\ \mathbf{1} & 0\end{bmatrix},\quad J^2=-\mathbf{1},\quad J=-J^{\mathrm{T}}. \tag{10.29}$$

Namely,

$$\partial_t V=JM_{\mathrm{Ht}}V,\quad V=\begin{bmatrix}\mathsf{p}_t\\ \mathsf{q}\end{bmatrix} \tag{10.30}$$

where

$$M_{\mathrm{Ht}}=\begin{bmatrix}\alpha^{-1} & -\alpha^{-1}\theta\partial_z\\ \partial_z\theta\alpha^{-1} & -\partial_z\theta\alpha^{-1}\theta\partial_z-\partial_z\eta\partial_z\end{bmatrix}= \tag{10.31}$$

$$=\begin{bmatrix}\mathbf{1} & 0\\ \partial_z\theta & \mathbf{1}\end{bmatrix}\begin{bmatrix}\alpha^{-1} & 0\\ 0 & -\partial_z\eta\partial_z\end{bmatrix}\begin{bmatrix}\mathbf{1} & -\theta\partial_z\\ 0 & \mathbf{1}\end{bmatrix}.$$

To obtain the first order equations with respect to z we pick the pair p_z,q. We use equations (10.25) and (10.20) for respectively $\partial_z\mathsf{p}_z$ and $\partial_z\mathsf{q}$. We eliminate p_t in (10.25) by using its representation (10.17) getting the system

$$\partial_z\mathsf{p}_z=-\partial_t\mathsf{p}_t=-\partial_t(\alpha\partial_t\mathsf{q}+\theta\partial_z\mathsf{q})=-\partial_t\alpha\partial_t\mathsf{q}-\partial_t\theta\eta^{-1}(\theta\partial_t\mathsf{q}-\mathsf{p}_z), \tag{10.32}$$

$$\partial_z\mathsf{q}=\eta^{-1}(\theta\partial_t\mathsf{q}-\mathsf{p}_z). \tag{10.33}$$

Observe that we used equation (10.33) to get the right-hand side of equation (10.32). The above system can be written as

$$\partial_z \begin{bmatrix} \mathsf{p}_z \\ \mathsf{q} \end{bmatrix} = \begin{bmatrix} \partial_t \theta \eta^{-1} & -\partial_t \alpha \partial_t - \partial_t \theta \eta^{-1} \theta \partial_t \\ -\eta^{-1} & \eta^{-1} \theta \partial_t \end{bmatrix} \begin{bmatrix} \mathsf{p}_z \\ \mathsf{q} \end{bmatrix}. \tag{10.34}$$

The system (10.34) can be transformed into the following canonical Hamiltonian form

$$\partial_z V = J M_{\mathrm{Hz}} V, \quad V = \begin{bmatrix} \mathsf{p}_z \\ \mathsf{q} \end{bmatrix}, \tag{10.35}$$

where

$$M_{\mathrm{Hz}} = \begin{bmatrix} -\eta^{-1} & \eta^{-1}\theta\partial_t \\ -\partial_t \theta \eta^{-1} & \partial_t \alpha \partial_t + \partial_t \theta \eta^{-1} \theta \partial_t \end{bmatrix} = \tag{10.36}$$
$$= \begin{bmatrix} \mathbf{1} & 0 \\ \partial_t \theta & \mathbf{1} \end{bmatrix} \begin{bmatrix} -\eta^{-1} & 0 \\ 0 & \partial_t \alpha \partial_t \end{bmatrix} \begin{bmatrix} \mathbf{1} & -\theta \partial_t \\ 0 & \mathbf{1} \end{bmatrix}.$$

Comparing expressions (10.31) and (10.36) we observe a noticeable difference in signs that is explained by the difference in signs in the expressions for the kinetic and potential energies in the Lagrangian density defined by (10.13).

We can transform the system (10.35)-(10.36) further yet into another form intimately related to the energy conservation law. For that we begin with the identity

$$M_{\mathrm{Hz}} = \begin{bmatrix} -\eta^{-1} & \eta^{-1}\theta\partial_t \\ -\partial_t \theta \eta^{-1} & \partial_t \alpha \partial_t + \partial_t \theta \eta^{-1} \theta \partial_t \end{bmatrix} = \tag{10.37}$$
$$= \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\partial_t \end{bmatrix} \begin{bmatrix} -\eta^{-1} & \eta^{-1}\theta \\ \theta\eta^{-1} & -\alpha - \theta\eta^{-1}\theta \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \partial_t \end{bmatrix}.$$

Based on (10.37), the system (10.35)-(10.36) can be recast into the following "Hamiltonian" form

$$\tilde{J} \partial_z V = \mathrm{i} \partial_t \tilde{M} V, \quad V = \begin{bmatrix} \mathsf{p}_z \\ \partial_t \mathsf{q} \end{bmatrix}, \tag{10.38}$$

where

$$\tilde{J} = \begin{bmatrix} 0 & \mathrm{i}\mathbf{1} \\ \mathrm{i}\mathbf{1} & 0 \end{bmatrix}, \qquad \tilde{M} = \begin{bmatrix} -\eta^{-1} & \eta^{-1}\theta \\ \theta\eta^{-1} & -\alpha - \theta\eta^{-1}\theta \end{bmatrix}. \tag{10.39}$$

When deriving the Hamiltonian equation (10.38)-(10.39) we used the following identity relating $\tilde{J}$ and J defined in (10.29)

$$\begin{bmatrix} \mathbf{1} & 0 \\ 0 & \partial_t \end{bmatrix} J \begin{bmatrix} \mathbf{1} & 0 \\ 0 & -\partial_t \end{bmatrix} = -\mathrm{i}\partial_t \begin{bmatrix} 0 & \mathrm{i}\mathbf{1} \\ \mathrm{i}\mathbf{1} & 0 \end{bmatrix} = -\mathrm{i}\partial_t \tilde{J} \tag{10.40}$$

Note that the matrices $\tilde{J}$ and $\tilde{M}$ are respectively antihermitian and hermitian, that is

$$\tilde{J}^* = -\tilde{J}, \qquad \tilde{M}^* = \tilde{M}. \tag{10.41}$$

Notice also that the definitions of V and $\tilde{J}$ in (10.38)-(10.39) imply the identity

$$V^* \tilde{J} V = \mathrm{i} \left[\mathsf{p}_z^* \partial_t \mathsf{q} + (\partial_t \mathsf{q})^* \, \mathsf{p}_z \right] = 2\mathrm{i} \operatorname{Re} \left\{ (\partial_t \mathsf{q})^* \, \mathsf{p}_z \right\}, \tag{10.42}$$

which via the theory of Hamiltonian equations can be associated with the energy conservation law as we show below.

10.3 Canonical and Hamilton equations

In this section we provide a concise review of canonical and Hamilton equations following [YakSta1, II.3.1-4]. By *canonical* we call an equation of the form

$$\tilde{J}\frac{dz}{dt}=\tilde{H}\left(t\right)z, \tag{10.43}$$

where $\tilde{H}\left(t\right)$ is a $2n\times 2n$ symmetric matrix valued function with real entries and $\tilde{J}$ is a constant $2n\times 2n$ nondegenerate skew-symmetric matrix with real entries, that is

$$\tilde{H}^{\mathrm{T}}\left(t\right)=\tilde{H}\left(t\right),\quad \tilde{J}^{\mathrm{T}}=-\tilde{J},\quad \left|\tilde{J}\right|\neq 0. \tag{10.44}$$

The matrix $\tilde{H}\left(t\right)$ in (10.43) is a called "Hamiltonian" of the equation. A standard form of $2n\times 2n$ nondegenerate skew-symmetric matrix J is

$$J_{2n}=\left[\begin{array}{cc}0&-\mathbf{1}_{n}\\ \mathbf{1}_{n}&0\end{array}\right]. \tag{10.45}$$

The canonical equation (10.43) can be always reduced to the special form

$$J_{2n}\frac{dx}{dt}=H\left(t\right)x, \tag{10.46}$$

by means of a linear change of variables, *i.e.*

$$x=Sz,\qquad \tilde{J}=S^{\mathrm{T}}J_{2n}S,\qquad \tilde{H}\left(t\right)=S^{\mathrm{T}}H\left(t\right)S \tag{10.47}$$

for some real nondegenerate $2n\times 2n$ matrix S.

We call an equation *Hamiltonian* if it is of the form (10.43) and (i) $\tilde{H}\left(t\right)$ is a Hermitian matrix with complex valued entries; (ii) $\tilde{J}$ is a constant nondegenerate antihermitian matrix, that is

$$\tilde{H}^{*}\left(t\right)=\tilde{H}\left(t\right),\qquad \tilde{J}^{*}=-\tilde{J},\qquad \left|\tilde{J}\right|\neq 0. \tag{10.48}$$

Canonical equations are of course Hamiltonian. A Hamiltonian equation (10.43) can be always reduced by a transformation $x=Sz$ with a nondegenerate S to the following special form

$$-iG_{0}\frac{dx}{dt}=H_{0}\left(t\right)x, \tag{10.49}$$

where $H_{0}\left(t\right)$ is a Hermitian matrix and

$$G_{0}=\left[\begin{array}{cc}\mathbf{1}_{p}&0\\ 0&-\mathbf{1}_{q}\end{array}\right],\ \text{where } p+q=2n. \tag{10.50}$$

Any matrix solution $Z\left(t\right)$ to the Hamiltonian equation (10.43) satisfies the identity, [YakSta1, II.3.4]

$$Z\left(t\right)^{*}\tilde{J}Z\left(t\right)=\tilde{J}, \tag{10.51}$$

and for any two vector solutions $z_{1}\left(t\right)$ and $z_{2}\left(t\right)$ there holds

$$\left(z_{1}\left(t\right),\tilde{J}z_{2}\left(t\right)\right)=\left[z_{1}\left(t\right)\right]^{*}\tilde{J}z_{2}\left(t\right)=\text{constant}\,. \tag{10.52}$$

(so called Poincaré invariant).

10.4 Energy exchange between subsystems

In this section we derive a general formula for the energy flux between two systems constituting a closed conservative system described by the Lagrangian $\mathcal{L} = \mathcal{L}(\mathsf{q}_t, \mathsf{q}_z)$ With the MTLB Lagrangian in mind let us put $\mathsf{q} = (Q, q)$ and assume that $\mathcal{L}$ can be split as

$$\mathcal{L} = \mathcal{L}_1 \left(\partial_t Q, \partial_{;z} Q\right) + \mathcal{L}_2 \left(\partial_t q, \partial_z q\right), \tag{10.53}$$

where

$$\partial_{;z} Q = \partial_z Q + B\partial_z q$$

and B is a fixed matrix. The Lagrangian $\mathcal{L}$ of the general form (10.53) describes two coupled interacting systems. The special form of coupling via the modified derivative $\partial_{;z}Q$ in (10.53) resembles the minimal coupling in the charge gauge theory. The variable q plays the role of the gauge field potential and B plays the role of coupling constant.

The corresponding Euler-Lagrange equations are (10.54), (10.55)

$$\partial_t \frac{\partial \mathcal{L}_1}{\partial \partial_t Q} + \partial_z \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} = 0, \tag{10.54}$$

$$\partial_t \frac{\partial \mathcal{L}_2}{\partial \partial_t q} + \partial_z \left[\frac{\partial \mathcal{L}_2}{\partial \partial_z q} + \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} B\right] = 0, \tag{10.55}$$

where the derivative $\frac{\partial \mathcal{L}}{\partial Q}$ of the scalar function $\mathcal{L}$ with respect to a column-vector Q is understood as a row-vector of the same dimension.

Recall now that the energy conservation law for the entire system has the form, [GelFom, 38.2-3], [Gold, 13.7]

$$\partial_t H + \partial_z S = 0, \tag{10.56}$$

where H and S are the energy and energy flux densities defined by

$$H = H_1 + H_2, \qquad S = S_1 + S_2, \tag{10.57}$$

with the following expressions for the individual energies and energy fluxes

$$H_1 = \frac{\partial \mathcal{L}_1}{\partial \partial_t Q} \partial_t Q - \mathcal{L}_1 \left(\partial_t Q, \partial_{;z} Q\right), \qquad S_1 = \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} \partial_t Q, \tag{10.58}$$

$$H_2 = \frac{\partial \mathcal{L}_2}{\partial \partial_t q} \partial_t q - \mathcal{L}_2 \left(\partial_t q, \partial_z q\right), \qquad S_2 = \left[\frac{\partial \mathcal{L}_2}{\partial \partial_z q} + \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} B\right] \partial_t q. \tag{10.59}$$

The above expressions imply the following identities for the first system

$$\begin{aligned} \partial_t H_1 = \frac{\partial \mathcal{L}_1}{\partial \partial_t Q} \partial_t^2 Q + \partial_t \left(\frac{\partial \mathcal{L}_1}{\partial \partial_t Q}\right) \partial_t Q - \frac{\partial \mathcal{L}_1}{\partial \partial_t Q} \partial_t^2 Q - \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} \left(\partial_{tz}^2 Q + B\partial_{tz}^2 q\right) = \\ = \partial_t \left(\frac{\partial \mathcal{L}_1}{\partial \partial_t Q}\right) \partial_t Q - \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} \left(\partial_{tz}^2 Q + B\partial_{tz}^2 q\right), \end{aligned} \tag{10.60}$$

$$\partial_z S_1 = \partial_z \left(\frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q}\right) \partial_t Q + \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} \partial_{tz}^2 Q. \tag{10.61}$$

The equations (10.60), (10.61) combined with the Euler-Lagrange equations (10.54) for the first system yield the following energy conservation law for the first system

$$\partial_t H_1 + \partial_z S_1 = -\frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} B \partial_{tz}^2 q, \tag{10.62}$$

where the right-hand side of (10.62) can be interpreted as the power flow density from the second system into the first one.

Carrying out similar computations for the second system we obtain

$$\begin{aligned} \partial_t H_2 &= \frac{\partial \mathcal{L}_2}{\partial \partial_t q} \partial_t^2 q + \partial_t \left(\frac{\partial \mathcal{L}_2}{\partial \partial_t q} \right) \partial_t q - \frac{\partial \mathcal{L}_2}{\partial \partial_t q} \partial_t^2 q - \frac{\partial \mathcal{L}_2}{\partial \partial_z q} \partial_{tz}^2 q = \\ &= \partial_t \left(\frac{\partial \mathcal{L}_2}{\partial \partial_t q} \right) \partial_t q - \frac{\partial \mathcal{L}_2}{\partial \partial_z q} \partial_{tz}^2 q, \end{aligned} \tag{10.63}$$

$$\partial_z S_2 = \partial_z \left[\frac{\partial \mathcal{L}_2}{\partial \partial_z q} + \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} B \right] \partial_t q + \left[\frac{\partial \mathcal{L}_2}{\partial \partial_z q} + \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} B \right] \partial_{tz}^2 q. \tag{10.64}$$

Combining equations (10.63) and (10.64) with the Euler-Lagrange equations (10.55) for the second system we obtain the following conservation law

$$\partial_t H_2 + \partial_z S_2 = \frac{\partial \mathcal{L}_1}{\partial \partial_{;z} Q} B \partial_{tz}^2 q, \tag{10.65}$$

where the right-hand side of (10.65) can be interpreted as the power density flow transferred from the first system into the second one.

Notice that relations (10.62) and (10.65) have right-hand sides of the same magnitude by with opposite signs. This can be viewed as a manifestation of the conservation of energy for the entire system. Indeed we recover (10.56) by adding (10.62) and (10.65).

10.5 Amplification for homogeneous MTLB systems: proofs.

This section contains rigorous formulations and proofs of the assertions made in Section 7.

Theorem 1 *Let L, C be symmetric, $n \times n$ positive matrices and let $u_0, \xi > 0$. Let $0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n$, be the eigenvalues of the positive symmetric matrix $L^{-1/2} C^{-1} L^{-1/2}$, repeated according to their multiplicity and let $v_i = \sqrt{\lambda_i}$. Let $\omega > 0$ be a fixed real number. Then, if either (i) $0 < u_0 \in (0, v_1]$ or (ii) $0 < u_0 \notin (0, v_1]$ and $\xi > 0$ is small enough, then there is a unique pair of complex conjugate solutions v_0, v_0^* of the equation $\left| \widetilde{A}(v) \right| = 0$, where $\widetilde{A}(v)$ is defined in (7.5), (7.6).*

Proof. By our assumption, the equation $|A(v)| = |-v^2 L + C^{-1}| = 0$ has exactly $2n$ real roots, $\pm v_1, \pm v_2, \ldots \pm v_n$, with $v_i > 0$ ($\lambda_i = v_i^2$). We assume in what follows that they are ordered: $0 < v_1 \leq v_2 \leq \ldots \leq v_n$ and each root is repeated a number of times equal to its multiplicity. If $|A(v)| \neq 0$, the following decomposition holds

$$\left| \widetilde{A}(v) \right| = |A(v)| \left[d - \xi (v - u_0)^2 - D^T (A(v))^{-1} D \right]. \tag{10.66}$$

This follows from the following more general fact: if M is a square block matrix of the form

$$M = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},$$

where A_1, A_4 are square matrices with $|A_1| \neq 0$, then

$$|M| = |A_1| \left| A_4 - A_3 A_1^{-1} A_2 \right|,$$

see e.g. [Bern, Lemma 2.8.6, page 108]. Observe that in our case $A_2 = D$ is a column matrix and $A_3 = D^{\mathrm{T}}$ is a row matrix. Then, if $|A(v)| \neq 0$, v is a root of $\left| \widetilde{A}(v) \right| = 0$ if and only if it is a root of the equation

$$-\xi(v - u_0)^2 = D^T (A(v))^{-1} D - d =: R(v).$$

The function $R(v)$ above turns out to have very nice properties. A well known fact from linear algebra concerning simultaneous diagonalization of two quadratic forms, one of which is positive, assures that there exists a non-degenerate matrix P such that

$$P^T A(v) P = \text{diag}\ (v) := \begin{bmatrix} v_1^2 - v^2 & 0 & \cdots & 0 \\ 0 & v_2^2 - v^2 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & v_n^2 - v^2 \end{bmatrix}.$$

Consequently,

$$D^T (A(v))^{-1} D = \widetilde{D}^T \begin{bmatrix} \frac{1}{v_1^2 - v^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{v_2^2 - v^2} & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \frac{1}{v_n^2 - v^2} \end{bmatrix} \widetilde{D} = \sum_1^n \frac{\widetilde{D}_i^2}{v_i^2 - v^2},$$

where $\widetilde{D} = P^T D$. Therefore,

$$R(v) = \sum_1^n \frac{\widetilde{D}_i^2}{v_i^2 - v^2} - d$$

is a rational function defined on the set $\{v : \ |A(v)| \neq 0\}$. It is immediately seen that R is an even function, increasing for $v > 0$ and decreasing for $v < 0$. Moreover, $\lim_{v \to \infty} R(v) = -d$ and the graph has vertical asymptotes at $v = \pm v_i$ if at least one of the $\widetilde{D}_k$ associated to v_i does not vanish (v_i may be a multiple root). Also,

$$R(0) + d = D^T (A(0))^{-1} D = D^T C D = \sum_{i=1}^n \sum_{j=1}^n C_{ij} D_i D_j =$$

$$= \sum_{i=1}^n \sum_{j=1}^n C_{ij} \left[\sum_{k=1}^n (C^{-1})_{ik} \right] \left[\sum_{r=1}^n (C^{-1})_{jr} \right] =$$

$$= \sum_{k=1}^n \sum_{r=1}^n \left[\sum_{i=1}^n (C^{-1})_{ik} \sum_{j=1}^n C_{ij} (C^{-1})_{jr} \right] = \sum_{k=1}^n \sum_{r=1}^n \left[\sum_{i=1}^n (C^{-1})_{ik} \delta_{ir} \right]$$

$$= \sum_{k=1}^n \sum_{r=1}^n (C^{-1})_{rk} = \sum_{k=1}^n D_k = d.$$

hence $R(0) = 0$. Since C^{-1} is non-degenerate, $D \neq 0$. Moreover, since the matrix P is non-degenerate, we have $\widetilde{D} \neq 0$. Therefore, the graph has at least two vertical asymptotes and always exhibits a central symmetric branch with the minimum at the point $(0,0)$. The number of real roots of the equation

$$-\xi(v-u_0)^2 = R(v)$$

is the number of intersection points of the parabola $y = -\xi(v-u_0)^2$ and the graph of R. For ξ small, it is exactly the number of monotonic branches (all branches, except for the central one), which coincides with the number of asymptotes. This number is always between 2 and $2n$, depending on the number of vanishing $\widetilde{D}_i$ and on the possible multiple roots; a precise description is given below. Moreover, it is easily seen that whenever $u_0 \in (0, v_1]$, the number of intersection points is equal to the number of asymptotes irrespective of the value of $\xi > 0$; see Figure 7.1 (a), whereas ξ small is needed otherwise; indeed, in Figure 7.1 (b) a large value of ξ produces three points of intersection with the far right branch of the graph of R, making the total number of intersection points exceed by two the number of asymptotes. If either (i) or (ii) holds, the intersections are transversal, hence the roots are simple. The previous assertions follow easily and rigorously from the monotonicity properties of R and f but their clear geometric meaning makes a lengthy proof unnecessary.

So far, we have considered the real roots of the equation $\left|\widetilde{A}(v)\right| = 0$ in the set $\{v : \det A(v) \neq 0\}$. Next, we consider the possible roots of the equation in the complementary set $\{\pm v_1, \pm v_2, ... \pm v_n\}$. Multiplying the matrix $\widetilde{A}(v)$ by $\widehat{P}^T$ from the left and by $\widehat{P}$ from the right, where

$$\widehat{P} = \begin{bmatrix} P & 0 \\ 0 & 1 \end{bmatrix},$$

there follows that the equation $\left|\widetilde{A}(v)\right| = 0$ is equivalent to the equation

$$\Delta(v) := \begin{vmatrix} v_1^2 - v^2 & 0 & .. & 0 & \\ 0 & v_2^2 - v^2 & .. & \vdots & \widetilde{D} \\ \vdots & \vdots & .. & 0 & \\ 0 & .. & 0 & v_n^2 - v^2 & \\ & \widetilde{D}^T & & & d - \xi(v-u_0)^2 \end{vmatrix} = 0,$$

where, as before, $\widetilde{D} = P^T D$. Let us analyze under what condition $\pm v_i$ are roots of the equation $\Delta(v) = 0$. Expanding the determinant with respect to the last column, and then the n-th order minor corresponding to $\widetilde{D}_i$ with respect to its i-th column, we get the expression

$$\Delta(v) = -\widetilde{D}_1^2 \begin{vmatrix} v_2^2 - v^2 & 0 & .. & 0 \\ 0 & v_3^2 - v^2 & .. & \vdots \\ \vdots & \vdots & .. & 0 \\ 0 & .. & 0 & v_n^2 - v^2 \end{vmatrix} - \widetilde{D}_2^2 \begin{vmatrix} v_1^2 - v^2 & 0 & .. & 0 \\ 0 & v_3^2 - v^2 & .. & \vdots \\ \vdots & \vdots & .. & 0 \\ 0 & .. & 0 & v_n^2 - v^2 \end{vmatrix} - ... \tag{10.67}$$

$$-\widetilde{D}_n^2 \begin{vmatrix} v_1^2 - v^2 & 0 & .. & 0 \\ 0 & v_2^2 - v^2 & .. & \vdots \\ \vdots & \vdots & .. & 0 \\ 0 & \cdots & 0 & v_{n-1}^2 - v^2 \end{vmatrix} + \left[d - \xi(v-u_0)^2\right] \begin{vmatrix} v_1^2 - v^2 & 0 & .. & 0 \\ 0 & v_2^2 - v^2 & .. & \vdots \\ \vdots & \vdots & .. & 0 \\ 0 & .. & 0 & v_n^2 - v^2 \end{vmatrix}.$$

We note in passing that the factorization (10.66) is easily obtained from the above expression by multiplying and dividing by the last determinant in the r.h.s. Assume first that $\pm v_i$ are simple roots of $|A(v)| = 0$, that is, that the binomial $v_i^2 - v^2$ appears only once in the matrix diag(v). Then, from the above expansion there follows that $\Delta(v_i) = 0$ if and only if $\widetilde{D}_i = 0$. Whenever this condition holds, the partial fraction $\widetilde{D}_i^2/(v_i^2 - v^2)$ in the expression of R disappears and the number of asymptotes is reduced by two. The number of real roots is thus increased by two ($\pm v_i$) and reduced by two, leaving the total number of roots unaffected.

Let us next consider the case of a multiple root. Assume that $v_i = v_{i+1} = ... = v_{i+k-1}$, hence the binomial $v_i^2 - v^2$ appears k times in diag(v), $k > 1$. Then $\pm v_i$ are necessarily roots of $\Delta(v) = 0$, as it can be readily seen from the above expansion. As for their multiplicity, there are two cases:

a) multiplicity $= k$, if all of $\widetilde{D}_i, \widetilde{D}_{i+1}, ...\widetilde{D}_{i+k-1}$ are zero, since in this case all non-zero terms in the expansion (10.67) contain k times the factor $v_i^2 - v^2$;

b) multiplicity $= k-1$, if not all of $\widetilde{D}_i, \widetilde{D}_{i+1}, ...\widetilde{D}_{i+k-1}$ are zero, since in this case the terms in (10.67) corresponding to the non-zero $\widetilde{D}_r$ with $r \in \{i, i+1, ...i+k-1\}$ contain the factor $v_i^2 - v^2$ only $k-1$ times, while the rest contain it k times.

In case (a), all the fractions with denominator $v_i^2 - v^2$ are missing in the rational function R, with consequent reduction of the number of asymptotes (with respect to the total possible number $2n$) by $2k$, which is precisely the number of additional roots, counting their multiplicity. Thus the total number of real roots is unaffected.

In case (b), there is only one fraction with denominator $v_i^2 - v^2$.Thus the total number of asymptotes is reduced by $2k-2$, which is the number of additional roots, counting their multiplicity.

Summing up, the total number of real roots of $\Delta(v) = 0$ (counting their multiplicity) is exactly $2n$ under our assumptions. Since the total number of roots of $\Delta(v) = 0$ is $2n+2$, there is necessarily one and only one pair of complex conjugate roots, thus proving the assertion.
■

The following theorem deals with the behavior of amplification as $\xi \to 0$ and as $\xi \to \infty$.

Theorem 2 *Let $v_0, \overline{v_0}$ with $\operatorname{Im} v_0 > 0$ denote the unique pair of complex conjugate roots of the equation $\left|\widetilde{A}(v)\right| = 0$ under the assumptions of Theorem 1. Let $k_0 = \omega/v_0$. Then,*

$$- \operatorname{Im} k_0 \sim \frac{C}{\sqrt{\xi}} \quad as\ \xi \to 0,\ C > 0. \tag{10.68}$$

Under the additional assumption $u_0 = v_1$ we also have

$$- \operatorname{Im} k_0 \sim \frac{C'}{\sqrt[3]{\xi}} \quad as\ \xi \to \infty,\ C' > 0. \tag{10.69}$$

Proof. The idea of the proof is to use very detailed information about real roots in combination with well known Vieta's formulas relating the roots to the coefficients of the corresponding polynomial. Let us first prove (10.68). Denote the $2n$ real roots of the equation $\left|\widetilde{A}(v)\right| = 0$ by v_1^+, v_2^+, ... v_n^+ ; $-v_1^-, -v_2^-, ... - v_n^-$, where $v_i^+, v_i^- > 0$ and $0 < v_1^+ \leq v_2^+ \leq ... \leq v_n^+$, $0 < v_1^- \leq v_2^- \leq ... \leq v_n^-$. The roots are repeated according to their multiplicity and some

of them may coincide with some v_i; see the proof of Theorem 1. If $n > 1$, the roots $v_i^+$and $-v_i^-$ with $i = 1, 2, ...(n-1)$ lie in the interval $[-v_n, v_n]$ for any value of $\xi > 0$ (recall that by v_i we denote the characteristic velocities of the MTL), whereas v_n^+ and $-v_n^-$, which correspond to the points of intersection of the parabola $y = -\xi(v - u_0)^2$ with the farthest right and farthest left branches of $y = R(v)$, lie outside of this very interval.

The extreme roots v_n^+ and $-v_n^-$ approach $+\infty$ (respectively $-\infty$) as $\xi \to 0$. This can be proved as follows: the parabola $y = -\xi(v-u_0)^2$ is decreasing for $v > u_0$, its intersection with the horizontal asymptote of R, $y = -d$, is $v^* = u_0 + \sqrt{d/\xi}$ and $R(v) < -d$ for $v > v_n$. Therefore, $v_n^+ > v^* \to +\infty$ as $\xi \to 0$. A similar argument can be applied to $-v_n^-$. In order to establish the asymptotic behavior of $\operatorname{Im} v_0$ we will make use of Vieta's formulas, relating the roots of a polynomial to its coefficients.

We start by observing that $\left|\widetilde{A}(v)\right|$ is a polynomial in v of degree $2n+2$:

$$\left|\widetilde{A}(v)\right| = a_{2n+2}v^{2n+2} + a_{2n+1}v^{2n+1} + ...a_1 v + a_{0.}$$

The coefficients a_{2n+2}, a_{2n+1} and a_0 can be easily computed in terms of the parameters. Indeed, $a_0 = \left|\widetilde{A}(0)\right|$, which can be computed by adding the first n rows of $\widetilde{A}(0)$ and subtracting the result from the last. Recalling that $D_i = \sum_j (C^{-1})_{ij}$ and that $d = \sum_i D_i$, we obtain

$$a_0 = \left|\widetilde{A}(0)\right| = \begin{vmatrix} C^{-1} & D \\ 0 & -\xi u_0^2 \end{vmatrix} = -\xi u_0^2 \left|C^{-1}\right|.$$

The only addend in $\left|\widetilde{A}(v)\right|$ yielding powers v^{2n+2} or v^{2n+1} is the diagonal one. Clearly, the diagonal term has the form

$$(-1)^n \prod_{i=1}^n L_{ii} v^{2n}\left[d - \xi(v-u_0)^2\right] + ... = (-1)^{n+1}\xi \prod_{i=1}^n L_{ii} v^{2n+2} + 2(-1)^n \xi u_0 \prod_{i=1}^n L_{ii} v^{2n+1} + ...$$

where the dots stand for lower order in v terms. Consequently,

$$a_{2n+2} = (-1)^{n+1}\xi \prod_{i=1}^n L_{ii}; \qquad a_{2n+1} = 2(-1)^n \xi u_0 \prod_{i=1}^n L_{ii}.$$

Vieta's formulas then imply

$$2 \operatorname{Re} v_0 + \sum_{i=1}^n v_i^+ - \sum_{i=1}^n v_i^- = -\frac{a_{2n+1}}{a_{2n+2}} = 2u_0 \tag{10.70}$$

$$(-1)^n |v_0|^2 \prod_{i=1}^n v_i^+ v_i^- = \frac{a_0}{a_{2n+2}} = (-1)^n \frac{u_0^2 \left|C^{-1}\right|}{\prod_{i=1}^n L_{ii}} \tag{10.71}$$

Next, we study the behavior as $\xi \to 0$ of both the sum and the product of the real roots. In the asymptotic formulas below, K_1, K_2, K_1', K_2' etc. denote positive constants depending on L, C, u_0 but not on ξ.

Let $n > 1$ and suppose that the graph of R has more than two asymptotes. First of all, we note that, as $\xi \to 0$, the parabola becomes flat and the roots $v_i^+, -v_i^-$ with $i = 1, 2, ...n-1$ become symmetric due to the symmetry of the graph of R. More precisely, if we denote by $\widehat{v_k^+}, -\widehat{v_k^-}$ with $k \in \{1, 2, ...n-1\}$ the abscissas of the points on the k-th right (respectively, k-th left) branch of the graph of R for which $R(\widehat{v_k^+}) = R(-\widehat{v_k^-}) = 0$, then clearly $v_k^+(\xi) \to \widehat{v_k^+}, v_k^-(\xi) \to \widehat{v_k^-}$ and $\widehat{v_k^+} = \widehat{v_k}^-$. Moreover, since R is strictly increasing for $v > 0$ and strictly decreasing for $v < 0$, $v_k^+(\xi) - \widehat{v_k^+} \sim -A_k\xi$, $-v_k^-(\xi) + \widehat{v_k^-} \sim B_k\xi$ as $\xi \to 0$, with $A_k, B_k > 0$.We also note the following fact, which is used in the proof of Section 8.1 and is a simple consequence of the lack of symmetry of the parabola $y = -\xi(v-u_0)^2$ with respect to the vertical axis: if $v_k^+, -v_k^-$ is a pair of real roots not belonging to the set $\{\pm v_1, \pm v_2, ... \pm v_n\}$ (and there is at least one such pair, see the proof of Theorem 1), then

$$v_k^+(\xi) - v_k^-(\xi) > 0. \tag{10.72}$$

Thus in particular $B_k > A_k$ in the above asymptotic relations.This inequality can be easily seen on the graph and given a simple analytical proof.

The roots belonging to the set $\{\pm v_1, \pm v_2, ... \pm v_n\}$ are symmetric and do not contribute to their sum. Therefore,

$$\sum_{i=1}^{n-1} v_i^+(\xi) - \sum_{i=1}^{n-1} v_i^-(\xi) = K_1\xi + o(\xi) \text{ as } \xi \to 0. \tag{10.73}$$

As for the product of roots, we have

$$\prod_{i=1}^{n-1} v_i^+(\xi) v_i^-(\xi) = (-1)^n K_2 + K_3\xi + o(\xi) \text{ as } \xi \to 0. \tag{10.74}$$

If there are only two asymptotes, then $v_1 = v_2 = ... = v_{n-1}$ and the left-hand side in (10.73) is zero. Also, the left-hand side in (10.74) is the constant $(-1)^n K_2$ Thus this case can be formally included in (10.73) and (10.74) by allowing K_1 and K_3 to vanish.

Let us now consider the extreme roots. As we noted, $v_n^+, v_n^- \to +\infty$. More precisely, since we have

$$-\xi(v_n^+ - u_0)^2 = R(v_n^+) \to -d \text{ as } \xi \to 0, \tag{10.75}$$

then necessarily $\lim_{\xi\to 0} \xi(v_n^+ - u_0)^2 = d > 0$ and thus

$$v_n^+(\xi) = \sqrt{\frac{d}{\xi}} + u_0 + E(\xi), \text{ where } E(\xi) = o\left(\sqrt{\frac{1}{\xi}}\right) \text{ as } \xi \to 0. \tag{10.76}$$

We need further refinement in the asymptotics of $E(\xi)$ as $\xi \to 0$. To this end, we recall that

$$R(v) + d \sim -\frac{A}{v^2} \text{ as } v \to \infty, \quad A > 0. \tag{10.77}$$

Plugging the expression (10.76) in (10.75) and using (10.77), we arrive at

$$2E(\xi)\sqrt{\frac{d}{\xi}} + E(\xi)^2 \to A \text{ as } \xi \to 0,$$

which implies

$$E(\xi) = K_3\sqrt{\xi} + o(\sqrt{\xi}) \ \text{ as } \xi \to 0.$$

Summing up, we have the following asymptotic representation for v_n^+ :

$$v_n^+(\xi) = \sqrt{\frac{d}{\xi}} + u_0 + K_3\sqrt{\xi} + o(\sqrt{\xi}) \ \text{ as } \xi \to 0. \tag{10.78}$$

An analogous representation takes place for v_n^- :

$$- v_n^-(\xi) = -\sqrt{\frac{d}{\xi}} + u_0 - K_3\sqrt{\xi} + o(\sqrt{\xi}) \ \text{ as } \xi \to 0. \tag{10.79}$$

Plugging (10.78), (10.79), (10.73) and (10.74) into (10.70) and (10.71) yields

$$\operatorname{Re} v_0 = o\left(\sqrt{\xi}\right) \ ; \quad |v_0|^2 = K_4\xi + o(\xi) \ \text{ as } \xi \to 0. \tag{10.80}$$

As a consequence,

$$\operatorname{Im} v_0 = \sqrt{K_4\xi + o(\xi)} = \sqrt{K_4}\sqrt{\xi} + o(\sqrt{\xi}) \text{ as } \xi \to 0$$

and, finally,

$$- \operatorname{Im} k_0 = \frac{\operatorname{Im} v_0}{|v_0|^2} \sim \frac{K_5}{\sqrt{\xi}} \text{ as } \ \xi \to 0,$$

thus proving (10.68) for $n > 1$. If $n = 1$, (10.78) and (10.79) hold and plugging into (10.70) and (10.71) yields the same result.

We turn now to the proof of (10.69), restricting ourselves to the case of just one line; the case of several lines can be handled in a similar fashion. First of all, it is clear that $v_1^+ \downarrow u_0$ and $-v_1^- \uparrow -u_0$ as $\xi \to \infty$. This can be rigorously proved in a way, similar to the above proof of the fact that $v_n^+ \uparrow \infty$, $v_n^- \downarrow -\infty$ as $\xi \to 0$. Put

$$v_1^+(\xi) = u_0 + G(\xi); \quad G(\xi) > 0, \quad G(\xi) \to 0 \text{ as } \xi \to \infty. \tag{10.81}$$

Near $u_0 = v_1$ we have

$$R(v) \sim \frac{A}{u_0 - v} \text{ as } v \to u_0^+ \ \text{ with } A > 0. \tag{10.82}$$

After use of (10.81) and (10.82), the equation

$$-\xi(v_1^+ - u_0)^2 = R(v_1^+)$$

yields the following asymptotic relation:

$$\xi G(\xi)^3 \sim A \text{ as } \xi \to \infty,$$

that is,

$$G(\xi) \sim \frac{K_1'}{\sqrt[3]{\xi}} \text{ as } \xi \to \infty.$$

An analogous formula takes place for the negative root:

$$v_1^-(\xi) = u_0 + H(\xi) \ \text{ with } H(\xi) \sim \frac{K_2'}{\sqrt[3]{\xi}} \text{ as } \xi \to \infty.$$

Therefore,

$$v_1^+(\xi) - v_1^-(\xi) \sim \frac{K_3'}{\sqrt[3]{\xi}} \text{ as } \xi \to \infty.$$

Applying again (10.70) and (10.71) we obtain

$$\operatorname{Re} v_0 = u_0 + \frac{K_4'}{\sqrt[3]{\xi}} + o\left(\frac{1}{\sqrt[3]{\xi}}\right), \qquad |v_0| \to 1/\sqrt{LC} + \frac{K_5'}{\sqrt[3]{\xi}} + o\left(\frac{1}{\sqrt[3]{\xi}}\right) \text{ as } \xi \to \infty. \quad (10.83)$$

Recall that $v_1 = 1/\sqrt{LC} = u_0$. The last two relations imply $\operatorname{Im} v_0 \sim K_6'/\sqrt[3]{\xi}$. Finally,

$$-\operatorname{Im} k_0 = \frac{\operatorname{Im} v_0}{|v_0|^2} \sim \frac{K_7'}{\sqrt[3]{\xi}} \qquad \text{as } \xi \to \infty.$$

as was to be proved. ■

One can also verify that if $u_0 < 1/\sqrt{LC}$ then both $\operatorname{Im} v_0$ and $\operatorname{Im} k_0$ have a finite, nonzero limit as $\xi \to \infty$.

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